Master Thesis

# The Well-Posedness for Semirelativistc Systems

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Submitted: 02/06/2014

# Abstract

The local well-posedness for the Cauchy problem of systems of two types of semirelativistic equations in one space dimension is shown.

The first system of semirelativistic equations is shown to be well-posed in the Sobolev space $H^s$ of order $s \geq 0$. We apply the standard contraction mapping theorem by using a Bourgain type space $X^{s,b}$. We also use an auxiliary space for solutions in the critical space $L^2 = H^0$. We show the conservation law of charge in the framework of Bourgain spaces without approximation procedure for local solutions. We give the global well-posedness by this conservation law and the persistence of regularity.

The second system of semirelativistic equations is shown to be global well-posed in the energy space $H^{1/2}$. In this case, the gain of regularity from the Bourgain method is not sufficient. Then, we apply the compactness argument with the energy conservation law. We use the charge and energy conservation laws to construct solutions. The continuous dependence of the solutions on time and initial data is also given.

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# Part I
# Introduction

We study time local and global well-posedness of the following Cauchy problem for a system of semirelativistic equations

$$\begin{cases} i\partial_t u \pm \sqrt{m_u^2 - \Delta}\; u = \lambda \overline{u} v, \\ i\partial_t v \pm \sqrt{m_v^2 - \Delta}\; v = \mu u^2, \\ (u(0), v(0)) = (u_0, v_0), \end{cases} \tag{NSR}$$

where $u, v$ are complex valued functions of $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\partial_t = \partial/\partial t$, $m_u, m_v \in \mathbb{R}$, $\lambda, \mu \in \mathbb{C}$, $\Delta = \partial_x^2 = (\partial/\partial x)^2$ is the Laplacian in $\mathbb{R}$, and $\overline{u}$ is the complex conjugate of $u$.

The purpose of this paper is to prove the global existence of finite charge or finite energy solutions to (NSR). To motivate our problem, we revisit three equations with regard to relativistic quantum mechanics: the Klein-Gordon, Dirac, and semirelativistic equation. The Klein-Gorden equation is the first model to describe relativistic quantum particles. However, since the Klein-Gordon equation is the second order in time, even the free Klein-Gorden equation does not admit a positive definite density. The Dirac equation is the second model of relativistic quantum particles. The free Dirac equation has the conserved probability density. However, the energy of the free Dirac equation is not positive definite for some initial data. The free semirelativistic equation avoid these definiteness problems – namely, it has the conserved probability density and energy whose sign is independent on the initial data. The semirelativistic equation is introduced form the Klein-Gordon equation by a formal factorization such as

$$\frac{\partial^2}{\partial t^2} + m^2 - \Delta = -\Big(i\frac{\partial}{\partial t} \pm \sqrt{m^2 - \Delta}\Big)\Big(i\frac{\partial}{\partial t} \mp \sqrt{m^2 - \Delta}\Big).$$

Since $\sqrt{1-\Delta}$ is nonlocal operator, the semirelativistic equation had not been studied well.

In these days, nonlinear semirelativistic equations has been studied in mathematics. For example, a semirelativistic equation with the Hartree type nonlinearity are regarded as a model of Boson stars and the well-posedness for the Cauchy problem has been studied. See [4,7,15] and the references therein. The Semirelativistic equation with power nonlinearity, which we study in this article, is regarded as a model of interactive relativistic quantum particles. Borgna and Rial studied the Cauchy problem for a single semirelativistic equation with cubic nonlinearity in [3] and they proved the existence of local solutions in $H^s$ with $s > 1/2$, where $H^s = (1 - \Delta)^{-s/2} L^2(\mathbb{R})$ is the usual Sobolev space. The method of their proof depends essentially on the Sobolev embedding $H^s \hookrightarrow L^\infty$. In the case where $s \le 1/2$, however, the method loses its meaning because the uniform control by $H^s$ norm breaks down. In the limiting case $s = 1/2$, a Vladimirov type argument [20, 21, 24] implies the uniqueness of weak solutions constructed by a compactness argument, see [14]. Meanwhile, we remark that Strichartz type estimates are not sufficient for a contraction argument unless the uniform control by $H^s$ norm is available. A similar situation happens in the case of nonlinear Dirac equations in space dimensions $n \ge 2$ [2, 6, 16–18]. We neither can not apply the Delgado-Candy trick which is the special technique for the Dirac equation in one dimension. This technique depends on algebraic structure of the Dirac equation to divided solutions into free solution part and uniform bounded part. However, the semirelativistic equation does not have the algebraic structure. See [1, 19].

Therefore, it is natural to introduce the Bourgain method or compactness argument to study (NSR) in $H^s$ with $0 \le s \le 1/2$. The system (NSR) is regarded as a semirelativistic approximation of the Schrödinger system

$$\begin{cases} i\partial_t u + \dfrac{\sigma_1}{2m_u}\Delta u = \lambda \overline{u} v, \\ i\partial_t v + \dfrac{\sigma_2}{2m_v}\Delta v = \mu u^2, \end{cases} \tag{NS}$$

where $\sigma_j \in \{-1, 1\}$. We refer the reader to [10–13] for recent results on the Cauchy problem for (NS). In the case of the Cauchy problem in $L^2 \times L^2$ for (NS), the signs of $\sigma_1$, $\sigma_2$ are not essential [12], while in this paper the combination $(\sigma_1, \sigma_2) = (1, -1)$ or $(-1, 1)$ is essential in (NSR) in connection with the quadratic interactions on the right hand sides as far as one tries to apply the Bourgain method in $L^2 \times L^2$. Because of this, we divide (NSR) into two cases:

$$\text{(NSR1)} \quad \begin{cases} i\partial_t u + \sqrt{m_u^2 - \Delta}\ u = \lambda \overline{u} v, \\ i\partial_t v - \sqrt{m_v^2 - \Delta}\ v = \mu u^2, \\ (u(0), v(0)) = (u_0, v_0). \end{cases}$$

and

$$\text{(NSR2)} \quad \begin{cases} i\partial_t u + \sqrt{m_u^2 - \Delta}\ u = \lambda \overline{u} v, \\ i\partial_t v + \sqrt{m_v^2 - \Delta}\ v = \mu u^2, \\ (u(0), v(0)) = (u_0, v_0). \end{cases}$$

We state our main results. For $a, b \in \mathbb{R}$, $a \vee b$ and $a \wedge b$ are the maximal and minimum, respectively. For norm spaces $X$ and $Y$, $(x, y) \in X \times Y$, and $z \in X \cap Y$,

$$\|(x, y) : X \times Y\| = \|x : X\| + \|y : Y\|, \qquad \|z : X \cap Y\| = \|z : X\| \vee \|z : Y\|.$$

**Theorem 1.** *Let $s \ge 0$ and let $(u_0, v_0) \in H^s \times H^s$. Then, there exists $T > 0$ and a unique pair of solutions $(u, v) \in C([0, T], H^s \times H^s)$ to (NSR1) . For this pair of solutions, we define the maximal existence time of solutions $T(s)$ as*

$$T(s) = T(u_0, v_0, s) = \sup \Big\{ T > 0\ ;\ \sup_{0<t<T} \big( \|u(t) : H^s\| + \|v(t) : H^s\| \big) < \infty \Big\}.$$

*Then, $T(s) = T(0)$.*

**Theorem 2.** *Let $\lambda$ and $\mu$ satisfy $\lambda = c\overline{\mu}$ with some constant $c > 0$. Let $s \ge 0$. Then, the solutions of Theorem 1 extend globally.*

**Theorem 3.** *Let $\lambda$ and $\mu$ satisfy $\lambda = c\overline{\mu}$ with some constant $c > 0$. The Cauchy problem (NSR2) has a unique pair of solutions $(u, v)$ in $C(\mathbb{R}, H^{1/2} \times H^{1/2}) \cap C^1(\mathbb{R}, H^{-1/2} \times H^{-1/2})$ for initial data $(u_0, v_0) \in H^{1/2} \times H^{1/2}$. In addition, let $(u_{0,n}, v_{0,n})_{n \in \mathbb{Z}_{\ge 0}}$ be a sequence in $H^{1/2} \times H^{1/2}$ which converges to $(u_0, v_0)$ in $H^{1/2} \times H^{1/2}$. For each $n$, let $(u_n, v_n)$ is the pair of solutions for* *the initial data* $(u_{0,n}, v_{0,n})$. *Then, for any* $T > 0$,

$$\left\|(u,v) - (u_n, v_n) : C([0,T], H^{1/2} \times H^{1/2})\right\| \to 0 \qquad \text{as} \qquad n \to \infty.$$

We introduce some notation to be used below. For $s \in \mathbb{R}$, $\dot{H}^s = (-\Delta)^{-s/2} L^2(\mathbb{R})$ is the homogeneous Sobolev space of order $s$. For a function $u$ of two variables (time and space), $\mathfrak{F}_x[u]$ denotes the Fourier transform with respect to the space variable $x$ and $\widetilde{u}$ denotes the Fourier transform with respect to the space-time variables. We also write $\hat{f}$ for the Fourier transform of a one-variable function $f$. For $m, t \in \mathbb{R}$, $U_m(t) = \exp[-it\sqrt{m^2 - \Delta}]$ denotes the free propagator for the semirelativistic equation

$$i\partial_t u - \sqrt{m^2 - \Delta} u = 0.$$

Then, the Cauchy problem for the single equation

$$i\partial_t u \mp \sqrt{m^2 - \Delta} u = f(u), \tag{SSR}$$

with initial data $u(0, \cdot) = u_0$ is rewritten in the form of the integral equation

$$u(t) = U_m(\pm t)u_0 - i \int_0^t U_m(\pm(t - t'))f(u(t'))dt',$$

where $f$ is a complex valued function. Let $(\cdot, \cdot)$ be the usual $L^2$ inner product. When $\lambda = c\overline{\mu}$ with a positive constant $c$, we define the charge $Q$ of (NSR) as

$$Q(u, v) = \left\|u : L^2\right\|^2 + c \left\|v : L^2\right\|^2.$$

The rest of this paper is organized as follows. In Part II, we show Theorems 1 and 2 by the Bourgain method. In Part III, we show Theorem 3 by a compactness argument. From Part IV, we collect recent works of the author and the supervisor.

**Part II**

# Study of (NSR1)

## 1 Introduction

In this part, we prove Theorems 1 and 2. Our proof is based on a Bourgain norm in $X^{s,b}$. We also use the auxiliary norm in $Y^s$ defined below especially for the critical case $s = 0$. We give several types of bilinear and trilinear estimates by means of those norms, which are applied to the arguments of the well-posedness and the persistence of regularity $T(s) = T(0)$. Particularly, we prove that the $H^s$ norms of the solutions never blow up before $L^2$ norms may blow up.

We also observe that it seems difficult to close our contraction mapping argument by using only $X^{s,b}$ norms in the critical case $s = 0$ and to apply this method to (NSR2) . We prove the fact that the bilinear estimate with $X^{s,b}$ norms fails in each case. If $s > 0$, we give a simpler proof which ensures the contraction argument depending exclusively on $X^{s,b}$ norms.

Under the constraint $\lambda = c\overline{\mu}$ with a positive constant $c$, we show the following conservation law of charge– namely, the conservation law of the $L^2$ norm :

$$Q(u(t), v(t)) = Q(u_0, v_0) \tag{II.1.1}$$

for any $t \in \mathbb{R}$. To prove (II.1.1), we apply the argument by one of us [22], which need not to take smooth approximations of solutions. Our proof uses only a weak regularity of solutions guaranteed in the corresponding Bourgain space. All the calculations for (II.1.1) make sense on the basis of the trilinear estimate given by Proposition 3.11 below. Then, we have the Theorem 2, since $T(s) = T(0)$ by Theorem 1.

For $m \geq 0$, $a, b \in \mathbb{R}$, $T_0 \in \mathbb{R}$, and $T > 0$, we define Bourgain norms

$$\|u : X^{s,b}_{m,\pm}\| = \left\| \langle \xi \rangle^s \left\langle \tau \pm \sqrt{m^2 + \xi^2} \right\rangle^b \widetilde{u}(\tau, \xi) : L^2_\tau L^2_\xi \right\|,$$

$$\begin{aligned} &\|u : X^{s,b}_{m,\pm}[T_0, T_0 + T]\| \\ &= \inf \left\{ \left\| u' : X^{s,b}_{m,\pm} \right\| \ ; \ \begin{array}{l} u'(t,x) = u(t,x) \text{ on } [T_0, T_0+T] \times \mathbb{R}, \\ \text{supp } u' \subset [T_0 - 2T, T_0 + 2T] \times \mathbb{R} \end{array} \right\}, \end{aligned}$$

$$\begin{aligned} &\|u : X'^{s,b}_{m,\pm}[T_0, T_0 + T]\| \\ &= \inf \left\{ \left\| u' : X^{s,b}_{m,\pm} \right\| \ ; u'(t,x) = u(t,x) \text{ on } [T_0, T_0+T] \times \mathbb{R} \right\}, \end{aligned}$$

where $\langle x \rangle = 1 + |x|$, and auxiliary norms

$$\|u : Y^{s}_{m,\pm}\| = \left\| \langle \xi \rangle^s \left\langle \tau \pm \sqrt{m^2 + \xi^2} \right\rangle^{-1} \widetilde{u} : L^2_\xi L^1_\tau \right\|,$$

$$\begin{aligned} &\|u : Y^{s}_{m,\pm}[T_0, T_0 + T]\| \\ &= \inf \left\{ \left\| u' : Y^{s}_{m,\pm} \right\| \ ; \ \begin{array}{l} u'(t,x) = u(t,x) \text{ on } [T_0, T_0+T] \times \mathbb{R}, \\ \text{supp } u' \subset [T_0 - 2T, T_0 + 2T] \times \mathbb{R} \end{array} \right\}. \end{aligned}$$

We note $\|u : X^{s,b}_{m,\pm}\| = \|\overline{u} : X^{s,b}_{m,\mp}\|$ and $\|u : Y^{s}_{m,\pm}\| = \|\overline{u} : Y^{s}_{m,\mp}\|$ for any $s, b, m \in \mathbb{R}$. We abbreviate these spaces as : $X^{s,b}_{\pm} = X^{s,b}_{0,\pm}$, $Y^{s}_{\pm} = Y^{s}_{0,\pm}$. In our proof, the following spaces are basic for the pair of solutions $(u, v)$:

$$\mathscr{X}^{s,b}[T_0, T_0 + T] = X^{s,b}_{-}[T_0, T_0 + T] \times X^{s,b}_{+}[T_0, T_0 + T],$$

$$\mathscr{X}'^{s,b}[T_0, T_0 + T] = X'^{s,b}_{-}[T_0, T_0 + T] \times X'^{s,b}_{+}[T_0, T_0 + T].$$

We use $\mathscr{X}^{s,b}[T_0, T_0 + T]$ for the proof of Theorems 1 and 2. The space $\mathscr{X}'^{s,b}[T_0, T_0 + T]$ is used for a simpler proof of the local well-posedness for $s > 0$ in Section 6. Let $\psi$ be a cut off function, namely, a smooth function with $0 \leq \psi \leq 1$, $\psi(t) = 1$ if $|t| \leq 1$ and $\psi(t) = 0$ if $|t| \geq 2$. For $T > 0$, $\psi_T(t) = \psi(T^{-1}t)$.

**Remark 1.1.** *For $s, b \geq 0$ , $T > 0$ and $m, T_0 \in \mathbb{R}$, function space $X^{s,b}_{m,\pm}[T_0, T_0 + T]$ is a quotient of a closed linear subspace of a weighted $L^2(\mathbb{R}^2)$ by another closed subspace. Since for $s, b \geq 0$, functions whose support is restricted on a subset of $\mathbb{R} \times \mathbb{R}$ compose a closed linear subspace on $X^{s,b}_{m,\pm}$ because $L^2(\mathbb{R}^2)$ is continuously embedded into them. Then, $\|\cdot : X^{s,b}_{m,\pm}[T_0, T_0 + T]\|$ is a quotient norm and $X^{s,b}_{m,\pm}[T_0, T_0{+}T]$ is a Banach space as long as for $s, b \geq 0$. However, we use the notation of $\|\cdot : X^{s,b}_{m,\pm}[T_0, T_0 + T]\|$ even if $b < 0$. We also use the notation $\|\cdot : Y^{s}_{m,\pm}[T_0, T_0 + T]\|$, even when $\|\cdot : Y^{s}_{m,\pm}[T_0, T_0 + T]\|$ is only a seminorm.*

We give a brief outline of the remainder of this part. We prepare the linear and bilinear estimates in Sections 2 and 3, respectively. The reason why the Bourgain method is not applicable to (NSR2) is also shown in Section 3. We give the proof of Theorem 1 in Section 4. We describe the proof of the $L^2$ conservation law (II.1.1) and Theorem 2 in Section 5. We give a simpler proof of the local existence in the case $s > 0$ and we show that the bilinear estimate fails with $s = 0$ in Section 6.

## 2 Linear Estimates

Here we collect some basic estimates. We consider the single equations

$$i\partial_t u \mp \sqrt{m^2 - \Delta}u = f(u), \tag{II.2.1}$$

where $u$ and $f$ are complex valued functions. The Cauchy problem for (II.2.1) with initial data $u(0, \cdot) = u_0$ is rewritten in the form of the integral equations

$$u(t) = U_m(\pm t)u_0 - i \int_0^t U_m(\pm(t - t'))f(u(t'))dt'.$$

To state the proof of our theorems, the following basic estimates are necessary.

**Lemma 2.1** ( [9, (2.19)]). *Let $m \in \mathbb{R}$. For any $s, b \geq 0$ and $u_0 \in H^s$,*

$$\|\psi(\cdot)U_m(\pm t)u_0 : X^{s,b}_{m,\pm}\| = \|\psi : H^b\|\|u_0 : H^s\|. \tag{II.2.2}$$

*In addition, for any* $0 < T < 1$,

$$\|\psi_T(\cdot)U_m(\pm t)u_0 : X^{s,1/2}_{m,\pm}\| \lesssim \|u_0 : H^s\|. \tag{II.2.3}$$

***proof***. The equality (II.2.2) is easily seen. The estimate (II.2.3) follows form scaling invariance of $\dot{H}^{1/2}$. Q.E.D.

**Proposition 2.2** ( [9, Lemma 2.1.]). *Let* $m \in \mathbb{R}$, $0 < T \le 1$ *and let* $s \ge 0$. *Then,*

$$\left\|\psi_T(\cdot)\int_0^{\cdot} U_m(\pm(\cdot - t'))F(t')\ dt' : X^{s,1/2}_{m,\pm}\right\| \lesssim \|F : X^{s,-1/2}_{m,\pm} \cap Y^s_{m,\pm}\| \tag{II.2.4}$$

*for* $F \in X^{s,-1/2}_{m,\pm} \cap Y^s_{m,\pm}$. *In addition, let* $\delta \ge 0$ *and* $b$ *satisfy* $-1/2 < b - 1 + \delta \le 0 \le b$. *Then,*

$$\left\|\psi_T(\cdot)\int_0^{\cdot} U_m(\pm(\cdot - t'))F(t')\ dt' : X^{s,b}_{m,\pm}\right\| \lesssim T^{\delta}\|F : X^{s,b-1+\delta}_{m,\pm}\| \tag{II.2.5}$$

*for* $F \in X^{s,b-1+\delta}_{m,\pm}$.

**Lemma 2.3** ( [9, Lemma 2.2.]). *Let* $m \in \mathbb{R}$. *If* $F \in Y^s_{m,\pm}$, *then* $\int_0^{\cdot} U_m(\cdot - t')F(t')dt' \in C(\mathbb{R} : H^s)$ *and it satisfies the estimate*

$$\left\|\int_0^{\cdot} U_m(\pm(\cdot - t'))F(t')dt' : C(\mathbb{R} : H^s)\right\| \lesssim \|F : Y^s_{m,\pm}\|.$$

To extract a positive power of $T$, we use the following lemma.

**Lemma 2.4** ( [9, Lemma 3.1.]). *Let* $s \in \mathbb{R}$, $0 \le b \le b'$, $T > 0$ *and let* $f \in X^{s,b'}_{\pm}$ *satisfy* $\mathrm{supp} f \subset [-T, T] \times \mathbb{R}$. *Then,*

$$\|f : X^{s,b}_{\pm}\| \lesssim T^{\gamma(b',b)}\|f : X^{s,b'}_{\pm}\|,$$

*where*

$$\gamma(b', b) = \begin{cases} b' - b & \text{if} \quad b' < 1/2, \\ b' - b + \varepsilon & \text{if} \quad b' = 1/2, \\ 1/2 - b/2b' & \text{if} \quad b' > 1/2 \end{cases}$$

*with* $\varepsilon > 0$ *sufficiently small.*

***proof***. By the Hölder inequality,

$$\begin{aligned}
&\left\|\langle\xi\rangle^s \langle\tau \pm |\xi|\rangle^b \widetilde{f} : L^2_\tau L^2_\xi\right\| \\
&\le \left\|\left\{\langle\xi\rangle^s \langle\tau \pm |\xi|\rangle^{b'} \widetilde{f}\right\}^{b/b'} : L^{2b'/b}_\tau L^{2b'/b}_\xi\right\| \left\|\left\{\langle\xi\rangle^s \widetilde{f}\right\}^{1-b/b'} : L^{2b'/(b'-b)}_\tau L^{2b'/(b'-b)}_\xi\right\| \\
&= \left\|f : X^{s,b'}_{\pm}\right\|^{b/b'} \left\|\langle\xi\rangle^s \widetilde{f} : L^2_\tau L^2_\xi\right\|^{1-b/b'}.
\end{aligned}$$

If $b' > 1/2$, then

$$\left\| \langle\xi\rangle^s \mathfrak{F}_x[f] : L_\xi^2 L_t^2 \right\| \lesssim T^{1/2} \| \langle\xi\rangle^s \mathfrak{F}_x[f] : L_\xi^2 L_t^\infty \| \\ \leq T^{1/2} \| \langle\xi\rangle^s \widetilde{f} : L_\xi^2 L_\tau^1 \| \\ \lesssim T^{1/2} \| f : X_\pm^{s,b'} \|.$$

Moreover, if $b' < 1/2$, then by the unitarity of $U_0$ and the Sobolev embedding,

$$\begin{aligned} &\left\| \langle\xi\rangle^s \mathfrak{F}_x[f] : L_t^2 L_\xi^2 \right\| \\ &= \| \langle\xi\rangle^s \mathfrak{F}_x[U_0(\pm t) f] : L_\xi^2 L_t^2 \| \\ &\lesssim T^{b'} \left\| \langle\xi\rangle^s \mathfrak{F}_x[U_0(\pm t) f] : L_\xi^2 L_t^{2/(1-2b')} \right\| \\ &\lesssim T^{b'} \| \langle\xi\rangle^s \mathfrak{F}_x[U_0(\pm t) f] : L_\xi^2 H_t^{b'} \| = T^{b'} \| f : X_\pm^{s,b'} \|. \end{aligned}$$

In the case $b' = 1/2$, for any $\varepsilon > 0$,

$$\| \langle\xi\rangle^s \mathfrak{F}_x[f] : L_t^2 L_\xi^2 \| \lesssim T^{1/2-\varepsilon} \left\| f : X_\pm^{s,1/2-\varepsilon} \right\| \leq T^{1/2-\varepsilon} \left\| f : X_\pm^{s,1/2} \right\|.$$

Q.E.D.

## 3 Bilinear and Trilinear Estimates

In this section, we derive nonlinear estimates for $X_{m,\pm}^{s,b}$ and $Y_{m,\pm}^s$ by the method originally proposed in [23]. Due to the next lemma, we may put $m_u = m_v = 0$ with respect to the Bourgain and auxiliary norms without loss of generality.

**Lemma 3.1.** *For any $m, M \in \mathbb{R}$, $X_{m,\pm}^{s,b} \simeq X_{M,\pm}^{s,b}$, $Y_{m,\pm}^s \simeq Y_{M,\pm}^s$ with equivalent norms.*

***proof.*** The lemma follows from the following inequality;

$$\begin{aligned} \frac{\left\langle \tau \pm \sqrt{m^2+\xi^2} \right\rangle}{\left\langle \tau \pm \sqrt{M^2+\xi^2} \right\rangle} &\leq 1 + \left| \frac{\left\langle \tau + \sqrt{m^2+\xi^2} \right\rangle - \left\langle \tau + \sqrt{M^2+\xi^2} \right\rangle}{\left\langle \tau \pm \sqrt{M^2+\xi^2} \right\rangle} \right| \\ &= 1 + \frac{\left| |\tau \pm \sqrt{m^2+\xi^2}| - |\tau \pm \sqrt{M^2+\xi^2}| \right|}{\left\langle \tau \pm \sqrt{M^2+\xi^2} \right\rangle} \\ &\leq 1 + |m - M| \end{aligned}$$

for any $\tau \in \mathbb{R}$. Q.E.D.

In addition, we need the following bilinear estimates for Sobolev norms.

**Lemma 3.2.** *Let $\alpha, \beta, \gamma \in \mathbb{R}$. Then, the inequality*

$$\|uv : H^{-\alpha}\| \lesssim \|u : H^\beta\| \|v : H^\gamma\|$$

*holds if and only if*

$$\alpha+\beta+\gamma \geq \frac{1}{2} \quad \text{and} \quad \alpha+\beta,\ \beta+\gamma,\ \gamma+\alpha > 0$$

*or*

$$\alpha+\beta+\gamma > \frac{1}{2} \quad \text{and} \quad \alpha+\beta,\ \beta+\gamma,\ \gamma+\alpha \geq 0.$$

**Lemma 3.3.** *Let $p \geq 1$ and let $\alpha,\ \beta,\ \gamma \geq 0$ satisfy $\alpha+\beta+\gamma > 1/p$. Then, there exists a positive constant $C$ such that the inequality*

$$\left\| \langle \tau+\delta_1 \rangle^{-\alpha} f * g : L^p_\tau \right\| \leq C \| \langle \tau+\delta_2 \rangle^{\beta} f : L^2_\tau \| \| \langle \tau+\delta_3 \rangle^{\gamma} g : L^2_\tau \|$$

*holds for any real numbers $\delta_1,\ \delta_2,\ \delta_3$ and any $f,\ g$ such that all the norms on the right hand side are finite.*

***proof***. By the Hölder and Young inequalities,

$$\begin{aligned} \left\| \langle \tau+\delta_1 \rangle^{-\alpha} f * g(\tau) : L^p_\tau \right\| &\lesssim \left\| f * g : L^{p(\alpha+\beta+\gamma)/(\beta+\gamma)} \right\| \\ &\lesssim \| \langle \tau+\delta_2 \rangle^{\beta} f(\tau) : L^2_\tau \| \| \langle \tau+\delta_3 \rangle^{\gamma} g(\tau) : L^2_\tau \| \end{aligned}$$

from which we obtain the lemma. Q.E.D.

**Lemma 3.4.** *Let $p$ and $\alpha$ satisfy $p \geq 1$ and $0 \leq \alpha \leq 1/p$. Let $\beta, \gamma, \kappa$ satisfy $0 \leq \beta, \gamma, \kappa \leq 1/2$ and $\alpha+\beta+\gamma+\kappa > 1/p+1/2$. Then, there exists a positive constant $C$ such that the inequality*

$$\begin{aligned} &\left\| \langle \tau+\delta_1 \rangle^{-\alpha} f * g * h : L^p_\tau \right\| \\ &\leq C \| \langle \tau+\delta_2 \rangle^{\beta} f : L^2_\tau \| \| \langle \tau+\delta_3 \rangle^{\gamma} g : L^2_\tau \| \| \langle \tau+\delta_4 \rangle^{\kappa} h : L^2_\tau \| \end{aligned}$$

*holds for any real numbers $\delta_1, \delta_2, \delta_3, \delta_4$ and any $f,\ g,\ h$ such that all the norms on the right hand side are finite.*

***proof***. Let $\epsilon = \alpha+\beta+\gamma+\kappa-1/p-1/2$. By the Hölder and the Young inequalities,

$$\begin{aligned} &\left\| \langle \tau+\delta_1 \rangle^{-\alpha} f * g * h : L^p_\tau \right\| \\ &\lesssim \| f * g * h : L^{p_1} \| \\ &\lesssim \| f : L^{p_2} \| \, \| g * h : L^{p_3} \| \\ &\lesssim \| f : L^{p_2} \| \, \| g : L^{p_4} \| \, \| h : L^{p_5} \| \\ &\lesssim \| \langle \tau+\delta_2 \rangle^{\beta} f : L^2_\tau \| \| \langle \tau+\delta_3 \rangle^{\gamma} g : L^2_\tau \| \| \langle \tau+\delta_4 \rangle^{\kappa} h : L^2_\tau \|, \end{aligned}$$

where

$$
\begin{aligned}
\frac{1}{p_1} &= \frac{1}{p} - \alpha + \frac{\alpha\varepsilon}{\alpha+\beta+\gamma+\kappa}, \\
\frac{1}{p_2} &= \frac{1}{2} + \beta - \frac{\beta\varepsilon}{\alpha+\beta+\gamma+\kappa}, \\
\frac{1}{p_3} &= \frac{1}{p_1} + 1 - \frac{1}{p_2}, \\
\frac{1}{p_4} &= \frac{1}{2} + \gamma - \frac{\gamma\varepsilon}{\alpha+\beta+\gamma+\kappa}, \\
\frac{1}{p_5} &= \frac{1}{2} + \kappa - \frac{\kappa\varepsilon}{\alpha+\beta+\gamma+\kappa}
\end{aligned}
$$

from which we obtain the lemma. Q.E.D.

For $s \geq 0$, we define $\lambda(s)$ as

$$
\lambda(s) = \begin{cases} 0 & \text{if} \quad s < 1/2, \\ s - 1/2 + \varepsilon & \text{if} \quad s \geq 1/2, \end{cases} \tag{II.3.1}
$$

where $\varepsilon > 0$ is sufficiently small. Here we state our main nonlinear estimates.

**Proposition 3.5.** *Let $s \geq 0$ and $0 \leq \rho < 1/2$. Then, the inequality*

$$
\begin{aligned}
&\left\| uv : X_+^{s,-1/2} \cap Y_+^s \right\| \\
&\lesssim \left\| u : X_-^{\lambda(s),1/2} \right\| \left\| v : X_-^{s,1/2-\rho} \right\| + \left\| u : X_-^{\lambda(s),1/2-\rho} \right\| \left\| v : X_-^{s,1/2} \right\|
\end{aligned} \tag{II.3.2}
$$

*holds for any $u \in X_-^{\lambda(s),1/2}$ and $v \in X_-^{s,1/2}$.*

We remark that the regularity $\lambda(s)$ in the both terms of $u$ on the right hand side is less than the regularity $s$ on the left hand side. Therefore, the estimate (II.3.2) with $s > 0$ does not follow directly from (II.3.2) with $s = 0$ and the Peetre's inequality: $\langle\xi\rangle^{s'} \lesssim (\langle\xi-\eta\rangle^{s'} + \langle\eta\rangle^{s'})$ for $s' \geq 0$. We can exchange the smoothness with respect to the space-time variables into the smoothness with respect to the space variable by using (II.3.3) from the nice combination of signs $\pm$ in (II.3.2). This technique is found in Lemma 5 of [23].

The symmetry inequality

$$
\begin{aligned}
&\left\| uv : X_-^{s,-1/2} \cap Y_-^s \right\| \\
&\lesssim \left\| u : X_+^{\lambda(s),1/2} \right\| \left\| v : X_+^{s,1/2-\rho} \right\| + \left\| u : X_+^{\lambda(s),1/2-\rho} \right\| \left\| v : X_+^{s,1/2} \right\|
\end{aligned}
$$

holds by (II.3.2) with taking complex conjugate of $u$ and $v$.

***proof***. It is enough to show

$$\begin{aligned}&\left\|uv : X_{+}^{s,-1/2}\right\| \\ &\lesssim \left\|u : X_{-}^{\lambda(s),1/2}\right\| \left\|v : X_{-}^{s,1/2-\rho}\right\| + \left\|u : X_{-}^{\lambda(s),1/2-\rho}\right\| \left\|v : X_{-}^{s,1/2}\right\|\end{aligned}$$

and

$$\left\|uv : Y_{+}^{s}\right\| \lesssim \left\|u : X_{-}^{\lambda(s),1/2}\right\| \left\|v : X_{-}^{s,1/2-\rho}\right\| + \left\|u : X_{-}^{\lambda(s),1/2-\rho}\right\| \left\|v : X_{-}^{s,1/2}\right\|.$$

Let

$$M(\tau,\xi,\sigma,\eta) = \Big|\tau + |\xi|\Big| \vee \Big|\tau - \sigma - |\xi - \eta|\Big| \vee \Big|\sigma - |\eta|\Big|.$$

Then, the triangle inequality implies

$$|\xi| + |\xi - \eta| + |\eta| \leq 3M(\tau,\xi,\sigma,\eta). \tag{II.3.3}$$

Also, we decompose the integral region as follows

$$\begin{aligned}A_1 &= \left\{(\tau,\xi,\sigma,\eta)\ ;\ M(\tau,\xi,\sigma,\eta) = \Big|\tau + |\xi|\Big|\right\}, \\ A_2 &= \left\{(\tau,\xi,\sigma,\eta)\ ;\ M(\tau,\xi,\sigma,\eta) = \Big|\tau - \sigma - |\xi - \eta|\Big|\right\}, \\ A_3 &= \left\{(\tau,\xi,\sigma,\eta)\ ;\ M(\tau,\xi,\sigma,\eta) = \Big|\sigma - |\eta|\Big|\right\}.\end{aligned}$$

(a) $X$ norm estimate with $s > 0$.

By the Minkowski inequality,

$$\begin{aligned}&\left\|\langle\xi\rangle^{s} \iint \langle\tau + |\xi|\rangle^{-1/2} \chi_{A_1}\ \widetilde{u}(\tau - \sigma, \xi - \eta)\ \widetilde{v}(\sigma,\eta)\ d\sigma d\eta : L^2_\tau L^2_\xi\right\| \\ &\lesssim \left\|\int \langle\xi\rangle^{s-1/2}\, I_1(\xi,\eta)\ d\eta : L^2_\xi\right\|,\end{aligned}$$

where

$$\begin{aligned}I_1(\xi,\eta) &= \left\|\int \left|\widetilde{u}(\tau - \sigma, \xi - \eta)\ \widetilde{v}(\sigma,\eta)\right|\ d\sigma : L^2_\tau\right\| \\ &\lesssim \left\|\langle\tau - |\xi - \eta|\rangle^{1/2-\rho}\,\widetilde{u}(\tau,\xi - \eta) : L^2_\tau\right\|\ \left\|\langle\tau - |\eta|\rangle^{1/2}\,\widetilde{v}(\tau,\eta) : L^2_\tau\right\|\end{aligned}$$

by Lemma 3.3. Since

$$\begin{aligned}&\frac{1}{2} - s + \lambda(s) + s \geq \frac{1}{2}, \\ &\frac{1}{2} - s + \lambda(s) > 0,\end{aligned}$$

and Lemma 3.2,

$$\left\| \int \langle \xi \rangle^{s-1/2} I_1(\xi,\eta)\ d\eta : L^2_\xi \right\| \lesssim \left\| u : X_-^{\lambda(s),1/2-\rho} \right\| \left\| v : X_-^{s,1/2} \right\|.$$

Similarly, for $j = 2, 3$,

$$\begin{aligned} &\left\| \langle \xi \rangle^{s} \iint \langle \tau + |\xi| \rangle^{-1/2} \chi_{A_j}\ \widetilde{u}(\tau-\sigma, \xi-\eta)\ \widetilde{v}(\sigma,\eta)\ d\sigma d\eta : L^2_\tau L^2_\xi \right\| \\ &\lesssim \left\| \int \langle \xi \rangle^{s-1/2} I_j(\xi,\eta)\ d\eta : L^2_\xi \right\|, \end{aligned}$$

where

$$\begin{aligned} I_2(\xi,\eta) &= \left\| \langle \tau + |\xi| \rangle^{-1/2} \int \langle \tau - \sigma - |\xi-\eta| \rangle^{1/2} \left| \widetilde{u}(\tau-\sigma,\xi-\eta)\ \widetilde{v}(\sigma,\eta) \right| d\sigma : L^2_\tau \right\| \\ &\lesssim \left\| \langle \tau - |\xi-\eta| \rangle^{1/2} \widetilde{u}(\tau,\xi-\eta) : L^2_\tau \right\| \ \left\| \langle \tau - |\eta| \rangle^{1/2-\rho} \widetilde{v}(\tau,\eta) : L^2_\tau \right\|, \end{aligned}$$

$$\begin{aligned} I_3(\xi,\eta) &= \left\| \langle \tau + |\xi| \rangle^{-1/2} \int \langle \sigma - |\eta| \rangle^{1/2} \left| \widetilde{u}(\tau-\sigma,\xi-\eta)\ \widetilde{v}(\sigma,\eta) \right| d\sigma : L^2_\tau \right\| \\ &\lesssim \left\| \langle \tau - |\xi-\eta| \rangle^{1/2-\rho} \widetilde{u}(\tau,\xi-\eta) : L^2_\tau \right\| \ \left\| \langle \tau - |\eta| \rangle^{1/2} \widetilde{v}(\tau,\eta) : L^2_\tau \right\| \end{aligned}$$

by Lemma 3.3. Then, for $j = 2, 3$, we obtain by Lemma 3.2

$$\begin{aligned} &\left\| \int \langle \xi \rangle^{s-1/2} I_j : L^2_\xi \right\| \\ &\lesssim \left\| u : X_-^{\lambda(s),1/2} \right\| \left\| v : X_-^{s,1/2-\rho} \right\| + \left\| u : X_-^{\lambda(s),1/2-\rho} \right\| \left\| v : X_-^{s,1/2} \right\|. \end{aligned}$$

(b) $X$ norm estimate with $s = 0$.

By Lemmas 3.2 and 3.3,

$$\begin{aligned} &\left\| uv : X_+^{0,-1/2} \right\| \\ &\lesssim \sum_{j=1}^{3} \left\| \int \langle \xi \rangle^{-1/4} \langle \eta \rangle^{-1/4} I_j(\xi,\eta)\ d\eta : L^2_\xi \right\| \\ &\lesssim \left\| u : X_-^{0,1/2} \right\| \left\| v : X_-^{0,1/2-\rho} \right\| + \left\| u : X_-^{0,1/2-\rho} \right\| \left\| v : X_-^{0,1/2} \right\|, \end{aligned}$$

where $I_1$, $I_2$ and $I_3$ are defined as in the case (a).

(c) $Y$ norm estimate with $s > 0$.

By the Minkowski inequality,

$$\begin{aligned}
&\left\| \langle \xi \rangle^s \iint \langle \tau + |\xi| \rangle^{-1} \chi_{A_1} \, \widetilde{u}(\tau - \sigma, \xi - \eta) \, \widetilde{v}(\sigma, \eta) \, d\sigma d\eta : L^2_\xi L^1_\tau \right\| \\
&\lesssim \left\| \int \langle \xi \rangle^{s-1/2} J_1(\xi, \eta) \, d\eta : L^2_\xi \right\|,
\end{aligned}$$

where

$$\begin{aligned}
J_1(\xi, \eta) &= \left\| \langle \tau + |\xi| \rangle^{-1/2} \int \left| \widetilde{u}(\tau - \sigma, \xi - \eta) \, \widetilde{v}(\sigma, \eta) \right| \, d\sigma : L^1_\tau \right\| \\
&\lesssim \left\| \langle \tau - |\xi - \eta| \rangle^{1/2-\rho} \, \widetilde{u}(\tau, \xi - \eta) : L^2_\tau \right\| \, \left\| \langle \tau - |\eta| \rangle^{1/2} \, \widetilde{v}(\tau, \eta) : L^2_\tau \right\|
\end{aligned}$$

by Lemma 3.3. Then, we obtain

$$\left\| \int \langle \xi \rangle^{s-1/2} J_1(\xi, \eta) \, d\eta : L^2_\xi \right\| \lesssim \left\| u : X_-^{\lambda(s), 1/2-\rho} \right\| \, \left\| v : X_-^{s, 1/2} \right\|$$

by Lemma 3.2. Similarly, for $j = 2, 3$,

$$\begin{aligned}
&\left\| \iint \langle \tau + |\xi| \rangle^{-1} \chi_{A_j} \, \widetilde{u}(\tau - \sigma, \xi - \eta) \, \widetilde{v}(\sigma, \eta) \, d\sigma d\eta : L^2_\xi L^1_\tau \right\| \\
&\lesssim \left\| \int \langle \xi \rangle^{s-1/2} J_j(\xi, \eta) \, d\eta : L^2_\xi \right\|,
\end{aligned}$$

where

$$\begin{aligned}
J_2(\xi, \eta) &= \left\| \langle \tau + |\xi| \rangle^{-1} \int \langle \tau - \sigma - |\xi - \eta| \rangle^{1/2} \left| \widetilde{u}(\tau - \sigma, \xi - \eta) \, \widetilde{v}(\sigma, \eta) \right| \, d\sigma : L^1_\tau \right\| \\
&\lesssim \left\| \langle \tau - |\xi - \eta| \rangle^{1/2} \, \widetilde{u}(\tau, \xi - \eta) : L^2_\tau \right\| \, \left\| \langle \tau - |\eta| \rangle^{1/2-\rho} \, \widetilde{v}(\tau, \eta) : L^2_\tau \right\|, \\
J_3(\xi, \eta) &= \left\| \langle \tau + |\xi| \rangle^{-1} \int \langle \sigma - |\eta| \rangle^{1/2} \left| \widetilde{u}(\tau - \sigma, \xi - \eta) \widetilde{v}(\sigma, \eta) \right| \, d\sigma : L^1_\tau \right\| \\
&\lesssim \left\| \langle \tau - |\xi - \eta| \rangle^{1/2-\rho} \, \widetilde{u}(\tau, \xi - \eta) : L^2_\tau \right\| \, \left\| \langle \tau - |\eta| \rangle^{1/2} \, \widetilde{v}(\tau, \eta) : L^2_\tau \right\|.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\left\| \int \langle \xi \rangle^{s-1/2} J_j(\xi, \eta) \, d\eta : L^2_\xi \right\| \\
&\lesssim \left\| u : X_-^{\lambda(s), 1/2} \right\| \, \left\| v : X_-^{s, 1/2-\rho} \right\| + \left\| u : X_-^{\lambda(s), 1/2-\rho} \right\| \, \left\| v : X_-^{s, 1/2} \right\|
\end{aligned}$$

from Lemma 3.2.

(d) $Y$ norm estimate with $s=0$.

By Lemmas 3.2 and 3.3,

$$\begin{aligned}
&\left\|uv : Y_+^0\right\| \\
&\lesssim \sum_{j=1}^{3} \left\| \int \langle\xi\rangle^{-1/4} \langle\eta\rangle^{-1/4} J_j(\xi,\eta)\, d\eta : L^2_\xi \right\| \\
&\lesssim \left\|u : X_-^{0,1/2}\right\| \left\|v : X_-^{0,1/2-\rho}\right\| + \left\|u : X_-^{0,1/2-\rho}\right\| \left\|v : X_-^{0,1/2}\right\|,
\end{aligned}$$

where $J_1$, $J_2$ and $J_3$ are defined as in the case (c).

Q.E.D.

**Remark 3.6.** *Proposition 3.5 is almost optimal. See Proposition 6.1 and Corollary 6.3.*

**Remark 3.7.** *The trick of exchanging smoothness is not applicable to the bilinear estimates $X_+^{s,b-1} \hookrightarrow X_+^{s,b} X_\pm^{s,b}$ and $X_-^{s,b-1} \hookrightarrow X_-^{s,b} X_\pm^{s,b}$ which one needs to use the Bourgain method for (NSR2) . In addition, the bilinear estimates $X_+^{s,b-1} \hookrightarrow X_+^{s,b} X_\pm^{s,b}$ and $X_-^{s,b-1} \hookrightarrow X_-^{s,b} X_\pm^{s,b}$ fail for $s \geq 1/2$ and any $b \in \mathbb{R}$. For any $s \leq 1/2$ and $b \in \mathbb{R}$, let $\widetilde{u}_\pm = \langle \tau \pm \xi\rangle^{-b-1} \langle\xi\rangle^{-s-1/2} \log\langle\xi\rangle^{-3/4}$. Then, $u_\pm \in X_\pm^{s,b}$ and*

$$\|u_+ u_\pm : X_+^{s,b}\| = \|u_- u_\pm : X_-^{s,b}\| = \infty.$$

*These estimates are calculated as follows:*

$$\begin{aligned}
&\|u_+u_+ : X_+^{s,b}\| \\
&= \Bigg\| \langle\xi\rangle^s \langle\tau + |\xi|\rangle^{b-1} \iint \langle\tau-\sigma+|\xi-\eta|\rangle^{-1} \langle\sigma+|\eta|\rangle^{-1} \\
&\qquad\qquad \cdot \langle\xi-\eta\rangle^{-s-1/2} \log\langle\xi-\eta\rangle^{-3/4} \langle\eta\rangle^{-s-1/2} \log\langle\eta\rangle^{-3/4} d\sigma d\eta : L^2 L^2 \Bigg\| \\
&\geq \Bigg\| \langle\xi\rangle^s \langle\tau+\xi\rangle^{b-1} \int_0^\xi \int_{\eta-1}^{\eta+1} \langle\tau-\sigma+\xi-\eta\rangle^{-1} \langle\sigma+\eta\rangle^{-1} \\
&\qquad\qquad \cdot \langle\xi-\eta\rangle^{-s-1/2} \log\langle\xi-\eta\rangle^{-3/4} \langle\eta\rangle^{-s-1/2} \log\langle\eta\rangle^{-3/4} d\sigma d\eta : L^2_{\xi\geq 2} L^2_{\xi-1\leq\tau\leq\xi+1} \Bigg\| \\
&\gtrsim \left\| \langle\xi\rangle^{-1/2} \log\langle\xi\rangle^{-3/4} \int_0^\xi \langle\eta\rangle^{-1} \log\langle\eta\rangle^{-3/4} d\eta : L^2_{\xi\geq 2} \right\| \\
&\gtrsim \left\| \langle\xi\rangle^{-1/2} \log\langle\xi\rangle^{-1/2} : L^2_{\xi\geq 2} \right\| = \infty,
\end{aligned}$$

$$\|u_+u_- : X_+^{s,b}\|$$
$$= \left\| \langle\xi\rangle^s\langle\tau+|\xi|\rangle^{b-1} \iint \langle\tau-\sigma+|\xi-\eta|\rangle^{-1}\langle\sigma-|\eta|\rangle^{-1} \cdot \langle\xi-\eta\rangle^{-s-1/2}\log\langle\xi-\eta\rangle^{-3/4}\langle\eta\rangle^{-s-1/2}\log\langle\eta\rangle^{-3/4} d\sigma d\eta : L^2L^2 \right\|$$
$$\geq \left\| \langle\xi\rangle^s\langle\tau+\xi\rangle^{b-1} \int_{-\xi}^{0}\int_{\eta-1}^{\eta+1} \langle\tau-\sigma+\xi-\eta\rangle^{-1}\langle\sigma+\eta\rangle^{-1} \cdot \langle\xi-\eta\rangle^{-s-1/2}\log\langle\xi-\eta\rangle^{-3/4}\langle\eta\rangle^{-s-1/2}\log\langle\eta\rangle^{-3/4} d\sigma d\eta : L^2_{\xi\geq 2}L^2_{\xi-1\leq\tau\leq\xi+1} \right\|$$
$$\gtrsim \left\| \langle\xi\rangle^{-1/2}\log\langle\xi\rangle^{-3/4}\int_0^\xi \langle\eta\rangle^{-1}\log\langle\eta\rangle^{-3/4} d\eta : L^2_{\xi\geq 2} \right\| = \infty,$$

*and the remainders are estimated similarly.*

**Corollary 3.8.** *Let $s \geq 0$, $0 \leq \rho < 1/2$ and let $T > 0$. Then,*

$$\left\| uv : X_{\pm}^{s,-1/2} \cap Y_{\pm}^{s} \right\| \lesssim T^\rho \left\| u : X_{\mp}^{\lambda(s),1/2} \right\| \left\| v : X_{\mp}^{s,1/2} \right\| \tag{II.3.4}$$

*for any $u \in X_{\mp}^{\lambda(s),1/2}$ and $v \in X_{\mp}^{s,1/2}$ such that* supp $u$, supp $v \subset [-T,T]\times\mathbb{R}$.

***proof***. By Proposition 3.5 and Lemma 2.4, we obtain (II.3.4).

Q.E.D.

The following bilinear estimate shall be used in Section 6.

**Proposition 3.9.** *Let $\varepsilon > 0$, $\rho \geq 0$, $b,\delta \in \mathbb{R}$ satisfy*

$$1+b-\delta > \frac{1}{2}+\varepsilon+\rho,$$
$$b+\delta+\varepsilon,\ \rho+\delta+\varepsilon \leq 1,$$
$$b-\varepsilon,\ b-\rho \geq 0,$$
$$s+\varepsilon \geq 1/2.$$

*Then,*

$$\left\| uv : X_{\pm}^{s,b-1+\delta} \right\| \lesssim \left\| u : X_{\mp}^{s,b} \right\| \left\| v : X_{\mp}^{s,b-\rho} \right\| + \left\| u : X_{\mp}^{s,b-\rho} \right\| \left\| v : X_{\mp}^{s,b} \right\| \tag{II.3.5}$$

*for any $u,v \in X_{\mp}^{s,b}$.*

***proof***. We use the same notation as in the proof of Proposition 3.5. Since $|\xi|, |\xi-\eta|, |\eta| \leq$

$3M(\tau,\xi,\sigma,\eta)$ and Lemma 3.3,

$$\begin{aligned}&\left\|\langle\xi\rangle^s \iint \langle\tau+|\xi|\rangle^{b-1+\delta}\,\chi_{A_1}\;\widetilde{u}(\tau-\sigma,\xi-\eta)\;\widetilde{v}(\sigma,\eta)\;d\sigma d\eta : L^2_\tau L^2_\xi\right\|\\ &\lesssim \left\|\int \langle\xi\rangle^{s-\varepsilon/3}\,\langle\xi-\eta\rangle^{-\varepsilon/3}\,\langle\eta\rangle^{-\varepsilon/3}\,K_1\;d\eta : L^2_\xi\right\|,\end{aligned}$$

where

$$\begin{aligned}K_1 &= \left\|\langle\tau+|\xi|\rangle^{b-1+\delta+\varepsilon}\int|\widetilde{u}(\tau-\sigma,\xi-\eta)\;\widetilde{v}(\sigma,\eta)|\;\;d\sigma : L^2_\tau\right\|\\ &\lesssim \left\|\langle\tau-|\xi-\eta|\rangle^{b-\rho}\,\widetilde{u}(\tau,\xi-\eta) : L^2_\tau\right\|\;\;\left\|\langle\tau-|\eta|\rangle^{b}\,\widetilde{v}(\tau,\eta) : L^2_\tau\right\|.\end{aligned}$$

Similarly, for $j=2,3$,

$$\begin{aligned}&\left\|\langle\xi\rangle^s \iint \langle\tau+|\xi|\rangle^{b-1+\delta}\,\chi_{A_j}\;\widetilde{u}(\tau-\sigma,\xi-\eta)\;\widetilde{v}(\sigma,\eta)\;d\sigma d\eta : L^2_\tau L^2_\xi\right\|\\ &\lesssim \left\|\int \langle\xi\rangle^{s-\varepsilon/3}\,\langle\xi-\eta\rangle^{-\varepsilon/3}\,\langle\eta\rangle^{-\varepsilon/3}\,K_j\;d\eta : L^2_\xi\right\|,\end{aligned}$$

where

$$\begin{aligned}K_2 &= \left\|\langle\tau+|\xi|\rangle^{b-1+\delta}\int\langle\tau-|\xi-\eta|\rangle^{\varepsilon}\,|\widetilde{u}(\tau-\sigma,\xi-\eta)\;\widetilde{v}(\sigma,\eta)|\;\;d\sigma : L^2_\tau\right\|\\ &\lesssim \left\|\langle\tau-|\xi-\eta|\rangle^{b}\,\widetilde{u}(\tau,\xi-\eta) : L^2_\tau\right\|\;\;\left\|\langle\tau-|\eta|\rangle^{b-\rho}\,\widetilde{v}(\tau,\eta) : L^2_\tau\right\|,\\ K_3 &= \left\|\langle\tau+|\xi|\rangle^{b-1+\delta}\int|\widetilde{u}(\tau-\sigma,\xi-\eta)\;\;\langle\tau-|\eta|\rangle^{\varepsilon}\,\widetilde{v}(\sigma,\eta)|\;\;d\sigma : L^2_\tau\right\|\\ &\lesssim \left\|\langle\tau-|\xi-\eta|\rangle^{b-\rho}\,\widetilde{u}(\tau,\xi-\eta) : L^2_\tau\right\|\;\;\left\|\langle\tau-|\eta|\rangle^{b}\,\widetilde{v}(\tau,\eta) : L^2_\tau\right\|.\end{aligned}$$

We obtain (II.3.5) by Lemma 3.2.

Q.E.D.

**Remark 3.10.** $b=1/2$ , $\delta=0$ , $\varepsilon=1/2$ *are the only numbers that ensures (II.3.5) for* $s=0$. *See Proposition 6.1.*

The next trilinear estimate shall be used to prove Theorem 2 in Section 5.

**Proposition 3.11.**

$$\begin{aligned}&\left\|\langle\tau\rangle^{-1}\,\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi) : L^1_\tau L^1_\xi L^1_\sigma L^1_\rho L^1_\eta\right\|\\ &\lesssim \left\|u : X^{0,1/2}_\pm\right\|\;\left\|v : X^{0,1/2}_\pm\right\|\;\left\|w : X^{0,1/2}_\pm\right\|\end{aligned}\tag{II.3.6}$$

*for any* $u,v,w\in X^{0,1/2}_\pm$.

***proof***. Let

$$N(\tau,\xi,\sigma,\rho,\varepsilon)=\Big|\tau\Big|\vee\Big|\sigma-\rho\pm|\xi-\eta|\Big|\vee\Big|\rho\pm|\eta|\Big|\vee\Big|\tau-\sigma\pm|\xi|\Big|.$$

Then, we have $|\xi|+|\xi-\eta|+|\eta|\leq 4N$. We also separate the integral region as follows

$$\begin{aligned}
B_1&=\Big\{(\tau,\sigma,\xi,\rho,\eta)\ ;N(\tau,\xi,\sigma,\rho,\varepsilon)=\Big|\tau\Big|\Big\},\\
B_2&=\Big\{(\tau,\sigma,\xi,\rho,\eta)\ ;N(\tau,\xi,\sigma,\rho,\varepsilon)=\Big|\sigma-\rho\pm|\xi-\eta|\Big|\Big\},\\
B_3&=\Big\{(\tau,\sigma,\xi,\rho,\eta)\ ;N(\tau,\xi,\sigma,\rho,\varepsilon)=\Big|\sigma\pm|\eta|\Big|\Big\},\\
B_4&=\Big\{(\tau,\sigma,\xi,\rho,\eta)\ ;N(\tau,\xi,\sigma,\rho,\varepsilon)=\Big|\tau-\sigma\pm|\xi|\Big|\Big\}.
\end{aligned}$$

By Lemmas 3.2, 3.4 and the Hölder inequality,

$$\begin{aligned}
&\left\|\chi_{B_1}\langle\tau\rangle^{-1}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi):L^1_\tau L^1_\xi L^1_\sigma L^1_\rho L^1_\eta\right\|\\
&\lesssim\Big\|\langle\tau\rangle^{-1/2}\langle\xi\rangle^{-1/4}\langle\eta\rangle^{-1/4}\\
&\qquad\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi):L^1_\tau L^1_\xi L^1_\sigma L^1_\rho L^1_\eta\Big\|\\
&\lesssim\Big\|\langle\xi\rangle^{-1/4}\langle\eta\rangle^{-1/4}\Big\|\langle\tau\pm|\xi-\eta|\rangle^{1/2}\widetilde{u}(\tau,\xi-\eta):L^2_\tau\Big\|\\
&\qquad\Big\|\langle\tau\pm|\eta|\rangle^{1/2}\widetilde{v}(\tau,\eta):L^2_\tau\Big\|:L^2_\xi(L^1_\eta)\Big\|\ \Big\|\langle\tau\pm|\xi|\rangle^{1/2}\widetilde{w}(\tau,\xi):L^2_\tau L^2_\xi\Big\|\\
&\lesssim\Big\|u:X^{0,1/2}_\pm\Big\|\ \Big\|v:X^{0,1/2}_\pm\Big\|\ \Big\|w:X^{0,1/2}_\pm\Big\|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left\|\chi_{B_2}\langle\tau\rangle^{-1}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi):L^1_\tau L^1_\xi L^1_\sigma L^1_\rho L^1_\eta\right\|\\
&\lesssim\Big\|\langle\tau\rangle^{-1}\langle\xi\rangle^{-1/4}\langle\eta\rangle^{-1/4}\langle\sigma-\rho\pm|\xi-\eta|\rangle^{1/2}\\
&\qquad\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi):L^1_\tau L^1_\xi L^1_\sigma L^1_\rho L^1_\eta\Big\|\\
&\lesssim\Big\|\langle\xi\rangle^{-1/4}\langle\eta\rangle^{-1/4}\Big\|\langle\tau\pm|\xi-\eta|\rangle^{1/2}\widetilde{u}(\tau,\xi-\eta):L^2_\tau\Big\|\\
&\qquad\Big\|\langle\tau\pm|\eta|\rangle^{1/2}\widetilde{v}(\tau,\eta):L^2_\tau\Big\|:L^2_\xi(L^1_\eta)\Big\|\ \Big\|\langle\tau\pm|\xi|\rangle^{1/2}\widetilde{w}(\tau,\xi):L^2_\tau L^2_\xi\Big\|\\
&\lesssim\Big\|u:X^{0,1/2}_\pm\Big\|\ \Big\|v:X^{0,1/2}_\pm\Big\|\ \Big\|w:X^{0,1/2}_\pm\Big\|.
\end{aligned}$$

The other integrations are estimated similarly. Q.E.D.

# 4 Proof of Theorem 1

We separate the proof for the existence and for the persistence of regularity.

### 4.1 Proof of Existence

Let $s \geq 0$, $(u_0, v_0) \in H^s \times H^s$ and let $0 < T \leq 1$. We define $\Phi : (u,v) \mapsto (\Phi_1(u,v),\ \Phi_2(u,v))$ as

$$\begin{cases} (\Phi_1(u,v))(t) = U_{m_u}(-t)u_0 - i\lambda \displaystyle\int_0^t U_{m_u}(t'-t)\ \overline{u(t')}\ v(t')dt', \\ (\Phi_2(u,v))(t) = U_{m_v}(t)v_0 - i\mu \displaystyle\int_0^t U_{m_v}(t-t')\ u(t')^2 dt'. \end{cases} \tag{II.4.1}$$

We also define a metric space

$$B^s(R,[0,T]) = \left\{(u,v) \in \mathscr{X}^{s,1/2}[0,T]\ ;\ \left\|(u,v) : \mathscr{X}^{s,1/2}[0,T]\right\| \leq R\right\}$$

with metric

$$\begin{aligned} d^s(\ (u_1,v_1),\ (u_2,v_2)\ ) &= \left\|(u_1,v_1) - (u_2,v_2) : \mathscr{X}^{s,1/2}[0,T]\right\| \\ &= \left\|u_1 - u_2 : X_-^{s,1/2}[0,T]\right\| + \left\|v_1 - v_2 : X_+^{s,1/2}[0,T]\right\|. \end{aligned}$$

We see $(B^s(R,[0,T]), d^s)$ is a complete metric space for any $s \geq 0$. We prove that $\Phi$ is a contraction map on $B^s(R,[0,T])$ for sufficiently large $R$ and sufficiently small $T$.

Let $(u,v) \in B^s(R,[0,T])$ and let $(u',v') \in X_-^{s,1/2} \times X_+^{s,1/2}$ satisfy

$$\begin{aligned} u' = u \ \text{ on } \ [0,T]\times\mathbb{R}, \quad \operatorname{supp} u' \subset [-2T,2T]\times\mathbb{R}, \\ v' = v \ \text{ on } \ [0,T]\times\mathbb{R}, \quad \operatorname{supp} v' \subset [-2T,2T]\times\mathbb{R}. \end{aligned}$$

Then, $\Phi_1(u,v)$ and $\Phi_2(u,v)$ are defined on $[0,T]\times\mathbb{R}$. Moreover,

$$\psi_T(t)\int_0^t U_{m_u}(t'-t)\ \overline{u'(t')}\ v'(t')\ dt' = \int_0^t U_{m_u}(t'-t)\ \overline{u(t')}\ v(t')dt',$$

$$\psi_T(t)\int_0^t U_{m_v}(t-t')\ u'(t')^2\ dt' = \int_0^t U_{m_v}(t-t')\ u(t')^2\ dt'$$

on $[0,T]\times\mathbb{R}$ and their supports are contained in $[-2T,2T]\times\mathbb{R}$. Then,

$$\begin{aligned} &\left\|\Phi_1(u,v) : X_-^{s,1/2}[0,T]\right\| \\ &\leq \left\|U_{m_u}(-t)\ u_0 : X_-^{s,1/2}[0,T]\right\| + \left\|\lambda\int_0^{\cdot} U_{m_u}(t'-\cdot)\ \overline{u(t')}\ v(t')dt' : X_-^{s,1/2}[0,T]\right\|. \end{aligned}$$

By Proposition 2.1,

$$\left\|U_{m_u}(-t)\ u_0 : X_-^{s,1/2}[0,T]\right\| \leq \left\|\psi_T(\cdot)U_{m_u}(-t)\ u_0 : X_-^{s,1/2}\right\| \lesssim \|u_0 : H^s\|.$$

By Proposition 2.2 and Corollary 3.8,

$$\begin{aligned}
&\left\| \int_0^{\cdot} U_{m_u}(t'-\cdot)\ \overline{u(t')}\ v(t')\ dt' : X_-^{s,1/2}[0,T] \right\| \\
&\quad \le \inf_{u',v'} \left\| \psi_T(\cdot) \int_0^{\cdot} U_{m_u}(t'-\cdot)\ \overline{u'(t')}\ v'(t')\ dt' : X_-^{s,1/2} \right\| \\
&\quad \lesssim \inf_{u',v'} \left\| \overline{u'}\ v' : X_-^{s,-1/2} \cap Y_-^s \right\| \\
&\quad \lesssim \inf_{u',v'} T^\rho \left\| u' : X_-^{s,1/2} \right\| \ \left\| v' : X_+^{s,1/2} \right\| \\
&\quad \lesssim T^\rho \left\| u : X_-^{s,1/2}[0,T] \right\| \ \left\| v : X_+^{s,1/2}[0,T] \right\| \le T^\rho R^2
\end{aligned}$$

for $0 < \rho < 1/2$. Similarly,

$$\begin{aligned}
&\left\| \Phi_2(u,v) : X_-^{s,1/2}[0,T] \right\| \\
&\le \left\| U_{m_v}(-t)\ v_0 : X_-^{s,1/2}[0,T] \right\| + \left\| \lambda \int_0^{\cdot} U_{m_v}(t'-\cdot)\ u(t')^2 dt' : X_-^{s,1/2}[0,T] \right\| \\
&\lesssim \|v_0 : H^s\| + T^\rho R^2.
\end{aligned}$$

This implies that $\Phi$ is a map from $B^s(R,[0,T])$ into itself for some $R$ and $T$. Moreover, let $(u_j,v_j) \in B^s(R,[0,T])$ for $j=1,2$ and let $(u_j',v_j') \in X_-^s \times X_+^s$ satisfy

$$\begin{aligned}
u_j' = u_j \ \text{ on } \ [0,T]\times\mathbb{R}, \quad \operatorname{supp} u_j' \subset [-2T,2T]\times\mathbb{R}, \\
v_j' = v_j \ \text{ on } \ [0,T]\times\mathbb{R}, \quad \operatorname{supp} v_j' \subset [-2T,2T]\times\mathbb{R}.
\end{aligned}$$

We have

$$\begin{aligned}
&\left\| \Phi_1(u_1,v_1) - \Phi_1(u_2,v_2) : X_-^{s,1/2}[0,T] \right\| \\
&\lesssim \inf_{u_1',u_2',v_1',v_2'} \left\{ \left\| \overline{(u_1'-u_2')}\ v_1' : X_-^{s,-1/2} \cap Y_-^s \right\| \right. \\
&\qquad\qquad \left. + \left\| \overline{u_2'}\ (v_1'-v_2') : X_-^{s,-1/2} \cap Y_-^s \right\| \right\} \\
&\le T^\rho \inf_{u_1'-u_2',v_1'} \left\{ \left\| v_1' : X_+^{s,1/2} \right\| \left\| u_1'-u_2' : X_-^{s,1/2} \right\| \right\} \\
&\quad + T^\rho \inf_{u_2',v_1'-v_2'} \left\{ \left\| u_2' : X_-^{s,1/2} \right\| \left\| v_1'-v_2' : X_+^{s,1/2} \right\| \right\} \\
&\lesssim T^\rho R \left\| (u_1,v_1) - (u_2,v_2) : \mathscr{X}^{s,1/2}[0,T] \right\|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left\|\Phi_2(u_1,v_1)-\Phi_2(u_2,v_2):X_+^{s,1/2}\right\| \\
&\lesssim T^\rho \inf_{u_1',u_2'}\left\|u_1'+u_2':X_-^{s,1/2}\right\|\left\|u_1'-u_2':X_-^{s,1/2}\right\| \\
&\lesssim T^\rho \left\|u_1+u_2:X_-^{s,1/2}[0,T]\right\|\left\|u_1-u_2:X_-^{s,1/2}[0,T]\right\| \\
&\lesssim T^\rho R\left\|(u_1,u_2)-(v_1,v_2):\mathscr{X}^{s,1/2}[0,T]\right\|.
\end{aligned}$$

Therefore $\Phi$ is a contraction map on $B^s(R,[0,T])$ with sufficiently small $T$.

## 4.2 Proof of Persistence Regularity

Let $s \geq 0$ and let $(u_0,v_0) \in H^s \times H^s$. By Theorem 1, we have the maximal existence time $T(s') > 0$ for $0 \leq s' \leq s$ such that there is a unique pair of local solutions $(u,v) \in C([0,T(s')),H^{s'} \times H^{s'})$. Since $s \geq \lambda(s)$, we have $T(s) \leq T(\lambda(s))$, where $\lambda(s)$ is as in (II.3.1). We show that if $T(s) < T(\lambda(s))$, then

$$\sup_{t\in[0,T(s))} \|(u,v)(t):H^s \times H^s\| < \infty, \tag{II.4.2}$$

namely, $T(s) = T(\lambda(s))$ from the point of view of a blow-up alternative argument. Let $T_1 = 1 \wedge \frac{T(\lambda(s))-T(s)}{2}$. For sufficiently large $C$, we define $R_1 > 0$ as follows

$$R_1 = 2C\left(1+\sup_{t\in[0,T(s)+T_1]}\|(u,v)(t):H^{\lambda(s)} \times H^{\lambda(s)}\|\right) < \infty.$$

We have $0 < T_2 < T_1$ such that for any $0 < T_0 < T(s)$ and any $0 < T < T_2$, $\Phi$ is a contraction map on $B^{\lambda(s)}(R_1,[T_0,T_0+T])$. Let $0 < \rho < 1/2$, and let $(u_j,v_j) \in B^{\lambda(s)}(R_1,[T_0,T_0+T])$. Let $u_j' \in X_-^{s,1/2}, u_j'' \in X_-^{\lambda(s),1/2}, v_j' \in X_+^{s,1/2}$, $v_j'' \in X_+^{\lambda(s),1/2}$ satisfy

$$\begin{aligned}
u_j' &= u_j \text{ on } [T_0,T_0+T]\times\mathbb{R}, &\quad \operatorname{supp} u_j' &\subset [T_0-2T,T_0+2T]\times\mathbb{R},\\
u_j'' &= u_j \text{ on } [T_0,T_0+T]\times\mathbb{R}, &\quad \operatorname{supp} u_j'' &\subset [T_0-2T,T_0+2T]\times\mathbb{R},\\
v_j' &= v_j \text{ on } [T_0,T_0+T]\times\mathbb{R}, &\quad \operatorname{supp} v_j' &\subset [T_0-2T,T_0+2T]\times\mathbb{R},\\
v_j'' &= u_j \text{ on } [T_0,T_0+T]\times\mathbb{R}, &\quad \operatorname{supp} v_j'' &\subset [T_0-2T,T_0+2T]\times\mathbb{R}
\end{aligned}$$

for $j = 1, 2$. Then, by Proposition 3.5

$$
\begin{aligned}
&\left\| \Phi_1(u_1, v_1) : X_-^{s,1/2}[T_0,\ T_0 + T] \right\| \\
&\le \left\| U_m(-t)u(T_0) : X_-^{s,1/2}[T_0,\ T_0 + T] \right\| \\
&\quad + \left\| \lambda \int_{T_0}^{\cdot} U_m(t' - \cdot)\overline{u_1(t')}\ v_1(t')\ dt'\ ;\ X_-^{s,1/2}[T_0,\ T_0 + T] \right\| \\
&\le C\|u(T_0) : H^s\| + CT^\rho \inf_{u_1'', v_1'} \left\| u_1'' : X_-^{\lambda(s),1/2} \right\| \left\| v_1' : X_+^{s,1/2} \right\| \\
&\le C\|u(T_0) : H^s\| + CT^\rho R_1 \left\| v_1 : X_+^{s,1/2}[T_0,\ T_0 + T] \right\|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left\| \Phi_2(u_1, v_1) : X_+^{s,1/2}[T_0,\ T_0 + T] \right\| \\
&\le \left\| U_M(t)v(T_0) : X_+^{s,1/2}[T_0,\ T_0 + T] \right\| + \left\| \mu \int_{T_0}^{\cdot} U_M(\cdot - t')u_1(t')^2\ dt'\ ;\ X_+^{s,1/2}[T_0,\ T_0 + T] \right\| \\
&\le C\|v(T_0) : H^s\| + CT^\rho \inf_{u_1', u_1''} \left\| u_1'' : X_-^{\lambda(s),1/2} \right\| \left\| u_1' : X_-^{s,1/2} \right\| \\
&\le C\|v(T_0) : H^s\| + CT^\rho R_1 \left\| u_1 : X_-^{s,1/2}[T_0, T_0 + T] \right\|.
\end{aligned}
$$

Let

$$R_2(T_0) = 2C\{1 + \|u(T_0) : H^s\| + \|v(T_0) : H^s\|\}$$

and let

$$T_3 = T_2 \wedge (8CR_1)^{-1/\rho} \wedge T(s) - T_0.$$

Then, for $0 < T < T_3$, $\Phi$ is a map on

$$B^{\lambda(s)}(R_1, [T_0, T_0 + T]) \cap B^s(R_2(T_0), [T_0, T_0 + T]).$$

In addition,

$$
\begin{aligned}
&\left\|\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2) : X_-^{s,1/2}[T_0,\ T_0+T]\right\| \\
&\le \left\|\lambda\int_{T_0}^{\cdot} U_m(t'-\cdot)\overline{u_1''(t')}\left\{v_1'(t')-v_2'(t')\right\}dt'\ ;\ X_-^{s,1/2}[T_0,\ T_0+T]\right\| \\
&\quad+\left\|\lambda\int_{T_0}^{\cdot} U_m(t'-\cdot)v_2''(t')\ \overline{\left\{u_1'(t')-u_2'(t')\right\}}\,dt'\ ;\ X_-^{s,1/2}[T_0,\ T_0+T]\right\| \\
&\le CT^{\rho}\inf_{u_1'',v_1'-v_2'}\left\|u_1'' : X_-^{\lambda(s),1/2}\right\|\left\|v_1'-v_2' : X_+^{s,1/2}\right\| \\
&\quad+CT^{\rho}\inf_{v_2'',u_1'-u_2'}\left\|v_2'' : X_+^{\lambda(s),1/2}\right\|\left\|u_1'-u_2' : X_-^{s,1/2}\right\| \\
&\le \frac{1}{4}\left\|(u_1,v_1)-(u_2,v_2) : \mathscr{X}^{s,1/2}[T_0,\ T_0+T]\right\|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left\|\Phi_2(u_1,v_1)-\Phi_2(u_2,v_2) : X_+^{s,1/2}[T_0,\ T_0+T]\right\| \\
&\le CT^{\rho}\inf_{u_1',u_1'',u_2',u_2''}\left\|u_1''+u_2'' : X_-^{\lambda(s),1/2}\right\|\left\|u_1'-u_2' : X_-^{s,1/2}\right\| \\
&\le \frac{1}{4}\left\|(u_1,v_1)-(u_2,v_2) : \mathscr{X}^{s,1/2}[T_0,T_0+T]\right\|.
\end{aligned}
$$

Therefore $\Phi$ is a contraction map and the pair of solutions $(u,v)$ is guaranteed in both $\mathscr{X}^{\lambda(s),1/2}[T_0,T_0+T]$ and $\mathscr{X}^{s,1/2}[T_0,T_0+T]$. If $T(s)-T_0<T_2\wedge(8CR_1)^{-1/\rho}$, then $T_3=T(s)-T_0$ and

$$\sup_{T\in[0,T(s)-T_0)}\|(u,v) : \mathscr{X}^{s,1/2}[T_0,\ T_0+T]\|\le R_2(T_0),$$

which together with Proposition 3.5 implies

$$\sup_{T\in[0,T(s)-T_0)}\left\{\|\overline{u}v : Y_-^s[T_0,\ T_0+T]\|+\|u^2 : Y_+^s[T_0,\ T_0+T]\|\right\}\le CR_2(T_0)^2.$$

Then, by lemma 2.3,

$$\sup_{t\in[T_0,T(s))}\|(u,v)(t) : H^s\times H^s\|\le C^2R_2(T_0)^2.$$

Thus, we obtain (II.4.2) and $T(s)=T(\lambda(s))=T(0)$.

# 5 Proof of the $L^2$ conservation and Theorem 2

In this section, we prove the $L^2$ conservation for Theorem 2.

Although we can justify a formal proof of the $L^2$ conservation by a smooth approximation argument, we give a different approach here. We derive the conservation laws without approximation of solution in the framework of the Bourgain method as we studied in the previous sections. For the Schrödinger equation, there is a proof of the conservation laws in the framework of the Strichartz estimate [22].

Let $(u_0, v_0) \in L^2 \times L^2$ and let $T > 0$ sufficiently small. Then, we have a pair of extensions $(u, v) \in X_-^{0,1/2} \times X_+^{0,1/2}$ of the solutions for the Cauchy problem (NSR1) such that for any $t \in [0, T]$,

$$u(t) = U_{m_u}(-t)u_0 - i\lambda \int_0^t U_{m_u}(t' - t)\overline{u}(t')v(t')dt',$$

$$v(t) = U_{m_v}(t)v_0 - ic^{-1}\overline{\lambda} \int_0^t U_{m_v}(t - t')u(t')^2 dt'.$$

Then,

$$\begin{aligned}
&\|u(t) : L^2\|^2 = \|U_m(t)u : L^2\|^2 \\
&= \left\| u_0 - i\lambda \int_0^t U_{m_u}(t')\overline{u}(t')v(t')dt' : L^2 \right\|^2 \\
&= \|u_0 : L^2\| - 2\mathrm{Im}\left( \hat{u}_0, \lambda \int_0^t \mathfrak{F}_x \left[U_{m_u}(t')\overline{u}(t')v(t')\right] dt' \right) \\
&\quad + \left\| \lambda \int_0^t \mathfrak{F}_x \left[U_{m_u}(t')\overline{u}(t')v(t')\right] dt' : L^2 \right\|^2,
\end{aligned}$$

where $(\cdot, \cdot)$ is the $L^2(\mathbb{R})$ inner product. We have

$$\int_0^t f(t')dt' = \int \frac{\exp[it\tau] - 1}{i\tau} \hat{f}(\tau)d\tau$$

for any $f \in L^1(\mathbb{R})$ such that $\hat{f} \in \langle \tau \rangle L^1_\tau(\mathbb{R})$. Moreover, the inequalities

$$\left\| \mathfrak{F}_x \left[\overline{u}v\right] : L^\infty_\xi L^1_t \right\| \leq \left\| u : L^2 L^2 \right\| \left\| v : L^2 L^2 \right\| \leq \left\| u : X_-^{0,1/2} \right\| \left\| v : X_+^{0.1/2} \right\|$$

hold by the Hölder inequality and

$$\iiiint \frac{\exp[it\cdot] - 1}{i\cdot} \overline{\widetilde{u}(\rho - \sigma, \eta - \xi)} \widetilde{v}(\rho, \eta) \overline{\widetilde{u}(\sigma - \cdot, \xi)} d\xi d\sigma d\eta d\rho \in L^1$$

by Proposition 3.11. Then,

$$
\begin{aligned}
&\left\|\lambda \int_0^t \mathfrak{F}_x\left[U_m(t')\overline{u}(t')v(t')\right]dt' : L^2\right\|^2 \\
&= 2\mathrm{Re}\int\int_0^t \lambda\mathfrak{F}_x[\overline{u}v](t')\,\overline{\lambda\mathfrak{F}_x\left[\int_0^{t'} U_m(t''-t')\overline{u}(t'')v(t'')dt''\right]}\,dt'd\xi \\
&= -2\mathrm{Im}\int\int_0^t \lambda\mathfrak{F}_x[\overline{u}v](t')\,\overline{\mathfrak{F}_x[U_m(-t')]\hat{u}_0 - \mathfrak{F}_x\left[u\right](t')}\,dt'd\xi \\
&= 2\mathrm{Im}\left(\hat{u}_0,\ \lambda\int_0^t \mathfrak{F}_x[U_m(t')\overline{u}v](t')dt'\right) \\
&\quad + 2\mathrm{Im}\ \lambda\iiiint\!\!\int \frac{\exp[it\tau]-1}{i\tau}\overline{\widetilde{u}(\rho-\sigma,\eta-\xi)}\widetilde{v}(\rho,\eta)\overline{\widetilde{u}(\sigma-\tau,\xi)}d\tau d\xi d\sigma d\eta d\rho.
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
&\|u(t) : L^2\|^2 - \|u_0 : L^2\|^2 \\
&= 2\mathrm{Im}\ \lambda\iiiint\!\!\int \frac{\exp[it\tau]-1}{i\tau}\overline{\widetilde{u}(\rho-\sigma,\eta-\xi)\widetilde{u}(\sigma-\tau,\xi)}\widetilde{v}(\rho,\eta)d\tau d\xi d\sigma d\eta d\rho.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\|v(t) : L^2\|^2 = \|U_M(-t)v : L^2\|^2 \\
&= \|v_0 : L^2\| - 2\mathrm{Im}\left(\hat{v}_0,\ c^{-1}\overline{\lambda}\int_0^t \mathfrak{F}_x\left[U_M(-t')u(t')^2\right]dt'\right) \\
&\quad + \left\|c^{-1}\overline{\lambda}\int_0^t \mathfrak{F}_x\left[U_M(-t')u(t')^2 dt'\right] : L^2\right\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|c^{-1}\overline{\lambda}\int_0^t \mathfrak{F}_x\left[U_m(-t')u(t')^2\right]dt' : L^2\right\|^2 \\
&= -2\mathrm{Im}\int\int_0^t c^{-1}\overline{\lambda}\mathfrak{F}_x[u(t')^2]\,\overline{\mathfrak{F}_x\left[ic^{-1}\overline{\lambda}\int_0^{t'} U_M(t'-t'')u(t'')^2dt''\right]}dt'd\xi \\
&= 2\mathrm{Im}\left(\hat{v}_0,\ c^{-1}\overline{\lambda}\int_0^t \mathfrak{F}_x\left[U_M(-t')u(t')^2\right]dt'\right) \\
&\quad + \frac{2}{c}\mathrm{Im}\ \overline{\lambda}\iiiint\!\!\int \frac{\exp[it\tau]-1}{i\tau}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)\overline{\widetilde{v}(\sigma-\tau,\xi)}d\tau d\xi d\sigma d\eta d\rho.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\|v(t) : L^2\|^2 - \|v_0 : L^2\|^2 \\
&= 2c^{-1}\mathrm{Im}\ \overline{\lambda}\iiiint\!\!\int \frac{\exp[it\tau]-1}{i\tau}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)\overline{\widetilde{v}(\sigma-\tau,\xi)}d\tau d\xi d\sigma d\eta d\rho.
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&- \mathrm{Im}\overline{\lambda} \iiiint\!\!\int \frac{\exp[it\tau]-1}{i\tau}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)\overline{\widetilde{v}(\sigma-\tau,\xi)}d\tau d\xi d\sigma d\eta d\rho \\
&= \mathrm{Im}\lambda \iiiint\!\!\int \frac{\exp[-it\tau]-1}{i(-\tau)}\overline{\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)}\ \widetilde{v}(\sigma-\tau,\xi)d\xi d\sigma d\eta d\rho d\tau \\
&= \mathrm{Im}\lambda \iiiint\!\!\int \frac{\exp[-it\tau]-1}{i(-\tau)}\overline{\widetilde{u}(\tau+\rho'-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)}\ \widetilde{v}(\rho',\xi)d\xi d\sigma d\eta d\rho' d\tau \\
&= \mathrm{Im}\lambda \iiiint\!\!\int \frac{\exp[it\tau']-1}{i\tau'}\overline{\widetilde{u}(\rho'-\sigma',\xi-\eta)\widetilde{u}(\sigma'-\tau',\eta)}\ \widetilde{v}(\rho',\xi)d\xi d\sigma' d\eta d\rho' d\tau' \\
&= \mathrm{Im}\lambda \iiiint\!\!\int \frac{\exp[it\tau']-1}{i\tau'} \\
&\qquad \times \overline{\widetilde{u}(\rho'-\sigma',\eta'-\xi')\widetilde{u}(\sigma'-\tau',\xi')}\ \widetilde{v}(\rho',\eta')\ d\xi' d\sigma' d\eta' d\rho' d\tau',
\end{aligned}
$$

where $\rho' = \sigma-\tau$, $\sigma' = \rho-\tau$, $\tau' = -\tau$, $\xi' = \eta$, and $\eta' = \xi$. Finally we have

$$
\|u(t) : L^2\|^2 + c\|v(t) : L^2\|^2 = \|u_0 : L^2\|^2 + c\|v_0 : L^2\|^2
$$

for $t \in [0,T]$.

## 6 Proof of Local Well-Posedness Independent of $Y$ Norm

In this section, we clarify why the auxiliary space $Y$ is important in our argument. We give an alternative proof of the existence of solutions for $s > 0$, without using the auxiliary norm $Y$. On the other hand, we shall explain why we need the norm $Y$ at least in our argument in the case where $s = 0$. It is important that $\delta(s)$ in this proof below is strictly positive. We exchange it into the positive power of $T$. Then, the contraction argument is completed when $T$ is sufficiently small.

***proof***. Let $s > 0$, $(u_0, v_0) \in H^s \times H^s$ and let $0 < T \le 1$. We take $b(s) = 3/4 \wedge (1+s)/2 > 1/2$ and $\delta(s) = 1/4 \wedge s/2 > 0$ for Proposition 3.9.

We define a metric space

$$
\begin{aligned}
&B'^s(R,T) \\
&= \left\{(u,v) \in \mathscr{X}'^{s,b(s)}[0,T]\ ;\ \left\|u : X_-'^{s,b(s)}[0,T]\right\| + \left\|v : X_+'^{s,b(s)}[0,T]\right\| \le R\right\}
\end{aligned}
$$

with metric

$$
d'(\ (u_1,v_1),\ (u_2,v_2)\ ) = \left\|(u_1,v_1)-(u_2,v_2) : \mathscr{X}'^{s,b(s)}[0,T]\right\|.
$$

We see $(B'^s(R,T), d')$ is a complete metric space. We prove that $\Phi$ defined as (II.4.1) is a contraction map on $B'^s(R,T)$ for sufficiently large $R$ and sufficiently small $T$.

Let $(u,v) \in B'^s(R,T)$ and let $(u',v') \in X_-^{s,b(s)} \times X_+^{s,b}$ satisfy

$$
u' = u \ \text{ on } \ [0,T]\times\mathbb{R}, \qquad v' = v \ \text{ on } \ [0,T]\times\mathbb{R}.
$$

We have

$$\begin{aligned}
&\left\| \Phi_1(u,v) : X'^{s,b(s)}_{-}[0,T] \right\| \\
&\leq \left\| U_m(-t)\ u_0 : X'^{s,b(s)}_{-}[0,T] \right\| \\
&\quad + \left\| \lambda \int_0^{\cdot} U_m(t'-t)\ \overline{u(t')}\ v(t') dt' : X'^{s,b(s)}_{-}[0,T] \right\|.
\end{aligned}$$

By Lemma 2.1,

$$\left\| U_m(-t)\ u_0 : X'^{s,b}_{-}[0,T] \right\| \leq \left\| \psi U_m(-t)\ u_0 : X'^{s,b}_{-} \right\| \lesssim \| u_0 : H^s \|.$$

By Propositions 2.2 and 3.9, we obtain

$$\begin{aligned}
&\left\| \int_0^{\cdot} U_m(t'-\cdot)\ \overline{u(t')}\ v(t')\ dt' : X'^{s,b(s)}_{-}[0,T] \right\| \\
&\leq \inf_{u',v'} \left\| \psi_T \int_0^{\cdot} U_m(t'-\cdot)\ \overline{u'(t')}\ v'(t')\ dt' : X^{s,b(s)}_{-} \right\| \\
&\lesssim \inf_{u',v'}\ T^{\delta(s)} \left\| \overline{u'}\ v' : X^{s,b(s)-1+\delta(s)}_{-} \right\| \\
&\lesssim \inf_{u',v'}\ T^{\delta(s)} \left\| u' : X^{s,b(s)}_{-} \right\| \ \left\| v' : X^{s,b(s)}_{+} \right\| \\
&\lesssim T^{\delta(s)} \left\| u : X^{s,b(s)}_{-}[0,T] \right\| \ \left\| v : X^{s,b(s)}_{+}[0,T] \right\| \leq T^{\delta(s)} R^2.
\end{aligned}$$

Similarly,

$$\left\| \Phi_2(u,v) : X^{s}_{+}[0,T] \right\| \lesssim \| v_0 : H^s \| + T^{\delta(s)} R^2.$$

Thus, $\Phi$ is a map from $B'^s(R,T)$ to $B'^s(R,T)$ for some $R$ and $T$. Moreover, let $(u_j, v_j) \in B'^s(R,T)$ for $j = 1,2$ and let $(u'_j, v'_j) \in X^{s,b(s)}_{-} \times X^{s,b(s)}_{+}$ satisfy

$$u'_j = u_j \ \text{ on } \ [0,T] \times \mathbb{R}, \qquad v'_j = v_j \ \text{ on } \ [0,T] \times \mathbb{R}.$$

Then, we have

$$\begin{aligned}
&\left\| \Phi_1(u_1,v_1) - \Phi_1(u_2,v_2) : X'^{s,b(s)}_{-}[0,T] \right\| \\
&\lesssim \inf_{u'_1,u'_2,v'_1,v'_2} T^{\delta(s)} \left\{ \left\| \overline{(u'_1 - u'_2)}\ v'_1 : X^{s,b(s)-1+\delta(s)}_{-} \right\| \right. \\
&\qquad\qquad \left. + \left\| \overline{u'_2}\ (v'_1 - v'_2) : X^{s,b(s)-1+\delta(s)}_{-} \right\| \right\} \\
&\lesssim T^{\delta(s)} \inf_{u'_1,u'_2,v'_1,v'_2} \left\{ \left\| v'_1 : X^{s,b(s)}_{+} \right\| \left\| u'_1 - u'_2 : X^{s,b(s)}_{-} \right\| \right. \\
&\qquad\qquad \left. + \left\| u'_2 : X^{s,b(s)}_{-} \right\| \left\| v'_1 - v'_2 : X^{s,b(s)}_{+} \right\| \right\} \\
&\lesssim T^{\delta(s)} R \left\| (u_1,v_1) - (u_2,v_2) : \mathscr{X}'^{s,b(s)}[0,T] \right\|.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\left\|\Phi_2(u_1,v_1)-\Phi_2(u_2,v_2) : X_+^{\prime s,b(s)}\right\| \\
&\lesssim T^{\delta(s)} \inf_{u_1',u_2'} \left\|u_1'+u_2' : X_-^{s,b(s)}\right\| \left\|u_1'-u_2' : X_-^{s,b(s)}\right\| \\
&\lesssim T^{\delta(s)} \left\|u_1+u_2 : X_-^{\prime s,b(s)}[0,T]\right\| \left\|u_1-u_2 : X_-^{\prime s,b(s)}[0,T]\right\| \\
&\lesssim T^{\delta(s)} R \left\|(u_1,u_2)-(v_1,v_2) : \mathscr{X}^{\prime s,b(s)}[0,T]\right\|.
\end{aligned}
$$

Thus, $\Phi$ is a contraction map on $B'^s(R,T)$ for sufficiently small $T$. Q.E.D.

The following proposition implies that we can not take $\delta > 0$ when $s = 0$ in the above proof.

**Proposition 6.1.** *For any $b \in [0,1/2) \cup (1/2,1]$, there exists a pair $(u,v) \in X_-^{0,b} \times X_-^{0,b}$ such that*

$$\left\|uv : X_+^{0,b-1}\right\| = \infty. \tag{II.6.1}$$

*Also for any $\delta > 0$, there exists a pair $(u,v) \in X_-^{0,1/2} \times X_-^{0,1/2}$ such that*

$$\left\|uv : X_+^{0,-1/2+\delta}\right\| = \infty. \tag{II.6.2}$$

**Remark 6.2.** *This is the reason why we use not only the norm $X_\pm^{s,b}$ but also the norm $Y_\pm^s$ and support restricted functions to obtain solutions of the Cauchy problem (NSR1) .*

**Proof of Proposition 6.1.** Suppose $1/2 < b \le 1$. Let $0 < 2\varepsilon \le b - 1/2$ and let

$$\widetilde{u_1}(\tau,\xi) = \widetilde{v_1}(\tau,\xi) = \langle\xi\rangle^{-\frac{1}{2}-\varepsilon} \langle\tau-|\xi|\rangle^{-b-1/2-\varepsilon}.$$

If $\tau > 2$, $\tau - 1 < \xi < \tau + 1$, then

$$
\begin{aligned}
&\langle\tau+|\xi|\rangle^{b-1} \\
&\quad \iint \langle\eta\rangle^{-\frac{1}{2}-\varepsilon} \langle\xi-\eta\rangle^{-1/2-\varepsilon} \langle\sigma-|\eta|\rangle^{-b-1/2-\varepsilon} \langle\tau-\sigma-|\xi-\eta|\rangle^{-b-1/2-\varepsilon} \, d\sigma d\eta \\
&\gtrsim \langle 2\tau+1\rangle^{b-1} \int_0^\xi \langle\eta\rangle^{-\frac{1}{2}-\varepsilon} \langle\xi-\eta\rangle^{-\frac{1}{2}-\varepsilon} \langle\tau-|\xi-\eta|-|\eta|\rangle^{-2b-2\varepsilon} \, d\eta \\
&\gtrsim \langle 2\tau+1\rangle^{b-1} \int_0^\xi (1+\xi+\eta(\xi-\eta))^{-1/2-\varepsilon} d\eta \\
&\gtrsim \langle 2\tau+1\rangle^{b-1} \int_0^\xi \langle\xi\rangle^{-1-2\varepsilon} \, d\eta \\
&\gtrsim \langle\tau+1\rangle^{-1/2}.
\end{aligned}
$$

This implies $u_1 v_1 \notin X_+^{0,b-1}$. Moreover, suppose $0 \le b < 1/2$. Let $b$ and $\delta$ satisfy $0 < 2\varepsilon \le 1/2 - b$

and let

$$\widetilde{u_2}(\tau,\xi) = \langle\xi\rangle^{-\frac{1}{2}-\varepsilon}\langle\tau-|\xi|\rangle^{-b-1/2-\varepsilon},$$
$$\widetilde{v_2}(\tau,\xi) = \langle\xi\rangle^{-\frac{1}{2}-\varepsilon}\langle\tau-|\xi|\rangle^{-b}\langle\tau+|\xi|\rangle^{-1/2-\varepsilon}.$$

Since for any real number $a$ and $b$, $\langle a+b\rangle \le \langle a\rangle\langle b\rangle$, for $\xi>0$,

$$\begin{aligned}
&\langle\tau+|\xi|\rangle^{b-1}\iint\langle\eta\rangle^{-\frac{1}{2}-\varepsilon}\langle\xi-\eta\rangle^{-1/2-\varepsilon}\langle\sigma-|\eta|\rangle^{-b-1/2-\varepsilon}\\
&\qquad\qquad\langle\tau-\sigma-|\xi-\eta|\rangle^{-b}\langle\tau-\sigma+|\xi-\eta|\rangle^{-1/2-\varepsilon}\,d\sigma d\eta\\
&\gtrsim\langle\tau+|\xi|\rangle^{b-1}\iint\langle\eta\rangle^{-\frac{1}{2}-\varepsilon}\langle\xi-\eta\rangle^{-b-1/2-\varepsilon}\langle\sigma-|\eta|\rangle^{-b-1/2-\varepsilon}\\
&\qquad\qquad\langle\tau-\sigma+|\xi-\eta|\rangle^{-b-1/2-\varepsilon}\,d\sigma d\eta\\
&\gtrsim\langle\tau+\xi\rangle^{b-1}\int_{-\infty}^{0}\langle\eta\rangle^{-\frac{1}{2}-\varepsilon}\langle\xi-\eta\rangle^{-b-\frac{1}{2}-\varepsilon}\langle\tau+\xi\rangle^{-2b-2\varepsilon}\,d\eta\\
&\gtrsim\langle\tau+\xi\rangle^{-b-1-2\varepsilon}\langle\xi\rangle^{-1/2}\notin L^2_{\xi>0}(L^2_\tau).
\end{aligned}$$

Therefore, $u_2v_2\notin X_+^{0,b-1}$. We complete the proof of (II.6.1).

Suppose $\delta>0$ and $b=1/2$. Let $\varepsilon$ satisfy $0<2\varepsilon\le\delta$ and let

$$\widetilde{u_3}(\tau,\xi)=\widetilde{v_3}(\tau,\xi)=\langle\xi\rangle^{-\frac{1}{2}-\varepsilon}\langle\tau-|\xi|\rangle^{-1-\varepsilon}.$$

If $\tau>2$, $\tau-1<\xi<\tau+1$, then

$$\begin{aligned}
&\langle\tau+|\xi|\rangle^{-1/2+\delta}\iint\langle\eta\rangle^{-\frac{1}{2}-\varepsilon}\langle\xi-\eta\rangle^{-1/2-\varepsilon}\langle\sigma-|\eta|\rangle^{-1-\varepsilon}\langle\tau-\sigma-|\xi-\eta|\rangle^{-1-\varepsilon}\,d\sigma d\eta\\
&\gtrsim\langle 2\tau+1\rangle^{-1/2+\delta}\int_0^{\xi}\langle\eta\rangle^{-\frac{1}{2}-\varepsilon}\langle\xi-\eta\rangle^{-\frac{1}{2}-\varepsilon}\langle\tau-|\xi-\eta|-|\eta|\rangle^{-1-2\varepsilon}\,d\eta\\
&\gtrsim\langle 2\tau+1\rangle^{-1/2+\delta}\int_0^{\xi}(1+\xi+\eta(\xi-\eta))^{-1/2-\varepsilon}d\eta\\
&\gtrsim\langle 2\tau+1\rangle^{-1/2+\delta}\int_0^{\xi}\langle\xi\rangle^{-1-2\varepsilon}\,d\eta\\
&\gtrsim\langle\tau+1\rangle^{-1/2}.
\end{aligned}$$

This yields $u_3v_3\notin X_+^{0,b-1}$ and we obtain (II.6.2). Q.E.D.

**Corollary 6.3.** *For any $b\in\mathbb{R}$ and $s<0$, there exists a pair $u,v\in X_-^{s,b}$ such that*

$$\left\|uv : X_+^{s,b-1}\right\| = \infty. \tag{II.6.3}$$

**Remark 6.4.** *Proposition 6.1 and Corollary 6.3 show that Proposition 3.5 is almost optimal.*

**Proof of Corollary 6.3.** Suppose $1/2 \le b \le 1$. Let $0 < \varepsilon < -s$ and let

$$\widetilde{u_1}(\tau,\xi) = \widetilde{v_1}(\tau,\xi) = \langle\xi\rangle^{-\frac{1}{2}-\varepsilon} \langle\tau - |\xi|\rangle^{-b-1/2-\varepsilon}.$$

If $\tau > 2$, $\tau - 1 < \xi < \tau + 1$, then

$$\begin{aligned}
&\langle\xi\rangle^{s} \langle\tau+|\xi|\rangle^{b-1} \\
&\quad \iint \langle\eta\rangle^{-s-\frac{1}{2}-\varepsilon} \langle\xi-\eta\rangle^{-s-1/2-\varepsilon} \langle\sigma-|\eta|\rangle^{-b-1/2-\varepsilon} \langle\tau-\sigma-|\xi-\eta|\rangle^{-b-1/2-\varepsilon} \, d\sigma d\eta \\
&\gtrsim \langle\xi\rangle^{s} \langle 2\tau+1\rangle^{b-1} \int_0^{\xi} \langle\eta\rangle^{-s-\frac{1}{2}-\varepsilon} \langle\xi-\eta\rangle^{-s-\frac{1}{2}-\varepsilon} \langle\tau-|\xi-\eta|-|\eta|\rangle^{-2b-2\varepsilon} \, d\eta \\
&\gtrsim \langle\xi\rangle^{s} \langle 2\tau+1\rangle^{b-1} \int_0^{\xi} (1+\xi+\eta(\xi-\eta))^{-s-1/2-\varepsilon} d\eta \\
&\gtrsim \langle 2\tau+1\rangle^{b-1} \int_0^{\xi} \langle\xi\rangle^{-2s-1-2\varepsilon} \, d\eta \\
&\gtrsim \langle\tau+1\rangle^{-1/2}.
\end{aligned}$$

This implies $u_1 v_1 \notin X_+^{s,b-1}$. Moreover, suppose $0 \le b < 1/2$. Let $b$ and $\delta$ satisfy $0 < 2\varepsilon \le 1/2 - b$ and let

$$\begin{aligned}
\widetilde{u_2}(\tau,\xi) &= \langle\xi\rangle^{-s-\frac{1}{2}-\varepsilon} \langle\tau-|\xi|\rangle^{-b-1/2-\varepsilon}, \\
\widetilde{v_2}(\tau,\xi) &= \langle\xi\rangle^{-s-\frac{1}{2}-\varepsilon} \langle\tau-|\xi|\rangle^{-b} \langle\tau+|\xi|\rangle^{-1/2-\varepsilon}.
\end{aligned}$$

Since for any real number $a$ and $b$, $\langle a+b\rangle \le \langle a\rangle \langle b\rangle$, for $\xi > 0$,

$$\begin{aligned}
&\langle\xi\rangle^{s} \langle\tau+|\xi|\rangle^{b-1} \iint \langle\eta\rangle^{-s-\frac{1}{2}-\varepsilon} \langle\xi-\eta\rangle^{-s-1/2-\varepsilon} \langle\sigma-|\eta|\rangle^{-b-1/2-\varepsilon} \\
&\qquad\qquad \langle\tau-\sigma-|\xi-\eta|\rangle^{-b} \langle\tau-\sigma+|\xi-\eta|\rangle^{-1/2-\varepsilon} \, d\sigma d\eta \\
&\gtrsim \langle\tau+|\xi|\rangle^{b-1} \iint \langle\eta\rangle^{-\frac{1}{2}-\varepsilon} \langle\xi-\eta\rangle^{-b-1/2-\varepsilon} \langle\sigma-|\eta|\rangle^{-b-1/2-\varepsilon} \\
&\qquad\qquad \langle\tau-\sigma+|\xi-\eta|\rangle^{-b-1/2-\varepsilon} \, d\sigma d\eta \\
&\gtrsim \langle\tau+\xi\rangle^{b-1} \int_{-\infty}^{0} \langle\eta\rangle^{-\frac{1}{2}-\varepsilon} \langle\xi-\eta\rangle^{-b-\frac{1}{2}-\varepsilon} \langle\tau+\xi\rangle^{-2b-2\varepsilon} \, d\eta \\
&\gtrsim \langle\tau+\xi\rangle^{-b-1-2\varepsilon} \langle\xi\rangle^{-1/2} \notin L^2_{\xi>0}(L^2_\tau).
\end{aligned}$$

Therefore, $u_2 v_2 \notin X_+^{0,b-1}$. We complete the proof of (II.6.3). Q.E.D.

## Part III

# Study of (NSR2)

## 1 Introduction

In this part, we show Theorem 3. As wee see Remark 3.7, the Bourgain method is not applicable to (NSR2) . We prove Theorem 3 by a compactness argument based on the energy and charge conservation of the solutions for (NSR2) . By these conservation laws, the uniformly boundedness in $H^{1/2}$ and equicontinuous in $L^2$ of a sequence of $H^1$ approximations for $H^{1/2}$ solution are given. We also use two convergence propositions. The first proposition is a kind of the Arzelà Ascoli theorem for Banach valued functions on $\mathbb{R}$. This proposition ensures a sequence of Banach valued functions converge weakly and the weak limits have the same continuity with the sequences. By the second proposition, the $H^{1/2}$ convergence of a $H^{1/2}$ sequence follows form the $L^2$ convergence and the convergence of their Energies. The existence of solutions follows from the first convergence proposition, where the weak limit of an approximation sequence is commutative with time differentials and the nonlinearities. The time continuity of solutions follows form both of the propositions. The continuity with respect to initial data follows from the second proposition and a Vladimirov argument. By the Vladimirov argument, we show the $L^2$ continuity with respect to initial data.

We define $((\cdot,\cdot),(\cdot,\cdot))_{\mathscr{H}} : (H^{1/2}\times H^{1/2})\times(H^{1/2}\times H^{1/2}) \to \mathbb{C}$ and $\|(\cdot,\cdot) : \mathscr{H}\| : H^{1/2}\times H^{1/2} \to \mathbb{R}$ as

$$\begin{aligned}\Big((u_1, v_1),\ (u_2, v_2)\Big)_{\mathscr{H}} &= \Big((m_u^2-\Delta)^{1/4}u_1,\ (m_u^2-\Delta)^{1/4}u_2\Big)_{L^2}\\ &\quad+\frac{c}{2}\Big((m_v^2-\Delta)^{1/4}v_1,\ (m_v^2-\Delta)^{1/4}v_2\Big)_{L^2},\\ \|(u,v):\mathscr{H}\| &= \sqrt{\Big((u,v),\ (u,v)\Big)_{\mathscr{H}}}.\end{aligned}$$

We also define the energy $E$ of (NSR2) and $M$ as follows:

$$\begin{aligned}E(u,v) &= \|(u,v):\mathscr{H}\|^2 - \mathrm{Re}(\lambda v, u^2)_{L^2},\\ M(u,v) &= \|(u,v):\mathscr{H}\| + Q(u,v) + \left\|(u,v) : L^2\times L^2\right\|.\end{aligned}$$

We give a brief outline of the remainder of this part. In Section2, some basic calculations are summarized and we prove Theorem 3 in Section 3.

## 2 Preliminary

Here, we collect some basic propositions. The Cauchy problem (NSR2) with initial data $(u(0), v(0)) = (u_0, v_0)$ is rewritten as the following system of the integral equations;

$$\begin{cases} u(t) = U_{m_u}(-t)u_0 - i\lambda \int_0^t U_{m_u}(t-t')\overline{u}(t')\ v(t')dt', \\ v(t) = U_{m_v}(-t)v_0 - ic^{-1}\overline{\lambda} \int_0^t U_{m_v}(t-t')u(t')^2 dt'. \end{cases} \tag{III.2.1}$$

For $s \geq 0$, $H^s$ solutions to (III.2.1) satisfy (NSR2) in $H^{s-1}$.

**Proposition 2.1.** *Let $(u_0, v_0) \in H^{1/2} \times H^{1/2}$. If there is a pair of solutions $(u, v) \in C(\mathbb{R}, H^{1/2} \times H^{1/2})$ to the integral equations* (III.2.1) *for the initial data $(u_0, v_0)$, then $Q(u(t), v(t)) = Q(u_0, v_0)$ for any $t$.*

***proof***.

$$\begin{aligned} \frac{d}{dt}\|u : L^2\|^2 &= 2\mathrm{Re}(\partial_t u, u) = 2\mathrm{Im}(i\partial_t u, u) \\ &= 2\mathrm{Im}(-\sqrt{m_u^2 - \Delta}\ u + \lambda\overline{u}v, u) \\ &= 2\mathrm{Im}(\lambda v, u^2), \\ \frac{d}{dt}\|v : L^2\|^2 &= 2\mathrm{Re}(\partial_t v, v) = 2\mathrm{Im}(i\partial_t v, v) \\ &= 2\mathrm{Im}(-\sqrt{m_v^2 - \Delta}\ v + c^{-1}\overline{\lambda}u^2, v) \\ &= -2c^{-1}\mathrm{Im}(\lambda v, u^2). \end{aligned}$$

Q.E.D.

**Proposition 2.2.** *Let $(u_0, v_0) \in H^{1/2} \times H^{1/2}$. If there is a pair of solutions $(u, v) \in C(\mathbb{R}, H^{1/2} \times H^{1/2}) \cap C^1(\mathbb{R}, H^{-1/2} \times H^{-1/2})$ to the integral equations* (III.2.1) *for the initial data $(u_0, v_0)$, then $E(u(t), v(t)) = E(u_0, v_0)$ for any $t$.*

***proof***. We assume $(u, v) \in C(\mathbb{R}, H^1 \times H^1)$. Then, $\|(m^2 - \Delta)^{1/4}u : L^2\|$ and $\|(m^2 - \Delta)^{1/4}v : L^2\|$ are differentiable and

$$\begin{aligned} &\frac{d}{dt}\|(m_u^2 - \Delta)^{\frac{1}{4}}u(t) : L^2\|^2 = 2\mathrm{Re}(-i\partial_t u(t) + \lambda\overline{u}v(t), \partial_t u(t)) = \mathrm{Re}(\lambda v(t), \partial_t(u^2)(t)), \\ &\frac{d}{dt}\|(m_v^2 - \Delta)^{\frac{1}{4}}v(t) : L^2\|^2 = 2\mathrm{Re}(-i\partial_t v + c^{-1}\overline{\lambda}u^2(t), \partial_t v(t)) = 2c^{-1}\mathrm{Re}(\partial_t(\lambda v)(t), u^2(t)). \end{aligned}$$

This shows the energy conservation for $H^1$ solution. We obtain the proposition by the smooth approximation. Q.E.D.

**Corollary 2.3.** *Let $(u_0, v_0) \in H^1 \times H^1$. For any $T > 0$, if there is a pair of solutions $(u, v) \in C([0, T], H^1 \times H^1)$ to the integral equations* (III.2.1) *for the initial data $(u_0, v_0)$, then $\|(u(t), v(t)) : H^{1/2} \times H^{1/2}\| \lesssim M(u_0, v_0)$ for any $t \in [0, T]$.*

***proof***. $H^{1/2} \times H^{1/2} = L^2 \times L^2 \cap \mathscr{H}$ and the $L^2 \times L^2$ norm of solutions is bounded by $Q(u_0, v_0)$. By Hölder and Minkowski inequalities,

$$\begin{aligned}\left|(\lambda v(t), u^2(t))_{L^2}\right| &\lesssim \left\|v(t) : L^2\right\| \left\|u(t) : L^4\right\|^2 \\ &\lesssim \left\|v(t) : L^2\right\| \left\|u(t) : L^2\right\| \left\|u(t) : \dot{H}^{1/2}\right\| \\ &\leq Q(u_0, v_0) \left\|(u, v)(t) : \mathscr{H}\right\|.\end{aligned}$$

Then,

$$\begin{aligned}&\|(u, v)(t) : \mathscr{H}\|^2 \\ &\leq \|(u_0, v_0) : \mathscr{H}\|^2 + \left|(\lambda v_0, u_0^2)_{L^2}\right| + \left|(\lambda v(t), u^2(t))_{L^2}\right| \\ &\lesssim \|(u_0, v_0) : \mathscr{H}\|^2 + Q(u_0, v_0)\|(u_0, v_0) : \mathscr{H}\| + Q(u_0, v_0)\|(u, v)(t) : \mathscr{H}\|.\end{aligned}$$

This shows $\|(u, v)(t) : \mathscr{H}\| \lesssim M(u_0, v_0)$. Q.E.D.

**Lemma 2.4.** *Let* $(u_0, v_0) \in H^1 \times H^1$. *Then, the integral equations* (III.2.1) *has a unique pair of global solutions* $(u, v) \in C(\mathbb{R}, H^1 \times H^1)$.

***proof***. By Sobolev embedding theorem, we have a unique pair of solutions $(u, v)$ in $C([0, T] : H^1 \times H^1)$ for sufficiently small $T > 0$. By Brezis-Galouet inequality, for any $f \in H^1$,

$$\|f : L^\infty\| \lesssim \|f : H^{1/2}\| \sqrt{\log\left(2 + \|f : H^1\| \,/\, \|f : H^{1/2}\|\right)}.$$

Moreover, for $a > 0$, $x^2 \log(2 + a/x)$ is increasing for $x$ on $\mathbb{R}_{>0}$. Indeed,

$$\begin{aligned}\frac{d}{dx}\, x^2 \log\left(2 + \frac{a}{x}\right) &= 2x \log\left(2 + \frac{a}{x}\right) - \frac{x^3}{2x + a}\frac{a}{x^2} \\ &\geq x(2\log 2 - 1) \geq 0.\end{aligned}$$

Then,

$$\begin{aligned}&\left\|(u, v)(t) : H^1 \times H^1\right\| \\ &\lesssim \left\|(u_0, v_0) : H^1 \times H^1\right\| \\ &+ \int_0^t \left\|u(t') : H^{1/2}\right\| \sqrt{\log\left(2 + \frac{\|u(t') : H^1\|}{\|u(t') : H^{1/2}\|}\right)} \left\|(u, v)(t') : H^1 \times H^1\right\| dt' \\ &+ \int_0^t \left\|v(t') : H^{1/2}\right\| \sqrt{\log\left(2 + \frac{\|v(t') : H^1\|}{\|v(t') : H^{1/2}\|}\right)} \left\|(u, v)(t') : H^1 \times H^1\right\| dt' \\ &\lesssim \left\|(u_0, v_0) : H^1 \times H^1\right\| \\ &+ M(u_0, v_0) \int_0^t \sqrt{\log\left(2 + \frac{\|(u, v)(t') : H^1 \times H^1\|}{M(u_0, v_0)}\right)} \left\|(u, v)(t') : H^1 \times H^1\right\| dt'.\end{aligned}$$

Suppose $a, b \geq 0$ and $f$ and $F$ be positive functions satisfy $f \leq F$ and

$$F(t) = a + b\int_0^t f(t')\sqrt{\log(f(t'))}dt'.$$

Then,

$$\frac{d}{dt}\sqrt{\log F(t)} = \frac{F'(t)}{F(t)\sqrt{\log F(t)}} \leq b.$$

This shows $\|(u, v)(t) : H^1 \times H^1\| < \infty$. By the blow-up alternative argument, the solutions can be extended globally. Q.E.D.

The next lemma plays an important role to construct solutions.

**Lemma 2.5.** *Suppose that $X$ and $Y$ are reflexive Banach spaces such that $Y \hookrightarrow X$ and $Y$ is dense in $X$. Let $K$ be a bounded closed ball of $Y$. Let $f$ be a non-negative valued function satisfy $f(0) = 0$ and $f$ is continuous at the origin. Let $(u_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence of $K$-valued function on $\mathbb{R}$ satisfy*

$$\sup_n \|u_n(t_1) - u_n(t_0) : X\| \leq f(t_1 - t_0). \tag{III.2.2}$$

*Then, we have a function $u : \mathbb{R} \to K$ and a subsequence of $(u_n)_{n\in\mathbb{Z}_{>0}}$ such that the subsequence converges to $u$ pointwisely in weak $Y$ topology. In addition, the weak limit $u$ satisfies that*

$$\|u(t_1) - u(t_0) : X\| \leq f(t_1 - t_0)$$

*and for any $l \in X^*$, $(\langle l, u_n\rangle_{X^*,X})_n$ converges uniformly on any bounded closed interval $I \subset \mathbb{R}$.*

***proof***. Put $\sigma$ be a bijective map from $\mathbb{Z}_{>0}$ to $\mathbb{Q}$. Since $(u_n(\sigma(1)))_{n\in\mathbb{Z}_{>0}} \subset K$, we can take $\rho(1, \cdot) : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $(u_{\rho(1,n)}(\sigma(1)))_{n\in\mathbb{Z}_{>0}}$ converges weakly in $Y$. Let $u(\sigma_1) \in Y$ be the weak limit of $(u_{\rho(1,n)}(\sigma(1)))_{n\in\mathbb{Z}_{>0}}$. Similarly, for any $k \in \mathbb{Z}_{>0}$, we can take $\rho(k, \cdot) : \mathbb{Z}_{>0} \to \rho(k-1, \mathbb{Z}_{>0})$ such that $(u_{\rho(k,n)}(\sigma(k)))_{n\in\mathbb{Z}_{>0}}$ converges weakly, and let $u(\sigma(k))$ be the weak limit of $(u_{\rho(k,n)}(\sigma(k)))_{n\in\mathbb{Z}_{>0}}$.

Put $v_n = u_{\rho(n,n)}$. We show $(v_n(t))_{n\in\mathbb{Z}_{>0}}$ converges weakly for any $t$. For the reflexivity of $Y$, it is enough to show $(\langle l, v_n(t)\rangle_{Y^*,Y})_{n\in\mathbb{Z}_{>0}}$ be a Cauchy sequence for any $l \in Y^*$. Let $\varepsilon > 0$ and let $l' \in X^*$ and $\sigma \in \mathbb{Q}$ satisfy $\|l - l' : Y^*\| \leq \varepsilon$ and

$$f(t-\sigma) \leq \frac{\varepsilon}{\|l' : X^*\| + 1}.$$

Then,

$$\begin{aligned}
&\left|\langle l, v_n(t) - v_m(t)\rangle_{Y^*,Y}\right| \\
&\leq \left|\langle l', v_n(t) - v_n(\sigma)\rangle_{X,X^*}\right| + \left|\langle l', v_m(t) - v_m(\sigma)\rangle_{X^*,X}\right| \\
&+ \left|\langle l', v_n(\sigma) - v_m(\sigma)\rangle_{X^*,X}\right| + \|l - l' : Y^*\| \ \mathrm{diam}(K) \\
&\leq 2\,\|l' : X^*\|\, f(t-\sigma) + \left|\langle l', v_n(\sigma) - v_m(\sigma)\rangle_{X^*,X}\right| + \mathrm{diam}(K)\varepsilon \\
&\leq (2 + \mathrm{diam}(K))\varepsilon + \left|\langle l', v_n(\sigma) - v_m(\sigma)\rangle_{X^*,X}\right|.
\end{aligned}$$

This means $(v_n(t))_{n\in\mathbb{Z}_{>0}}$ converges weakly in $Y$ for any $t \in \mathbb{R}$. Let $u(t) \in K$ be the weak limit of $(v_n(t))_{n\in\mathbb{Z}_0}$. The continuity in $X$ of $u$ is obtained by

$$\begin{aligned}
&\left|\langle l, u(t_1) - u(t_0)\rangle_{X^*,X}\right| \\
&\leq \left|\langle l, u(t_1) - v_n(t_1)\rangle_{X^*,X}\right| + \left|\langle l, u(t_0) - v_n(t_0)\rangle_{X^*,X}\right| \\
&+ \left|\langle l, v_n(t_1) - v_n(t_0)\rangle_{X^*,X}\right| \\
&\leq \|l : X^*\|\, f(t_1 - t_0) + \left|\langle l, u(t_1) - v_n(t_1)\rangle_{X^*,X}\right| \\
&+ \left|\langle l, u(t_0) - v_n(t_0)\rangle_{X^*,X}\right| \\
&\to \|l : X^*\|\, f(t_1 - t_0) \qquad \text{as} \quad n \longrightarrow \infty.
\end{aligned}$$

Moreover, let $\varepsilon > 0$ and $(t_m)_{m=0}^M$ be an increasing sequence such that $t_0 = \min I$ and $t_M = \max I$ and $\sup_m f(t_{m+1} - t_m) < \varepsilon$. Let $N$ be sufficient large such that

$$\left|\langle l, v_n(t_m) - u(t_m)\rangle_{X^*,X}\right| < \varepsilon$$

for any $0 \leq m \leq M$ and $n \geq N$. Then, for $n \geq N$,

$$\begin{aligned}
&\left|\langle l, v_n(t) - u(t)\rangle_{X^*,X}\right| \\
&< \left|\langle l, v_n(t_m) - u(t_m)\rangle_{X^*,X}\right| + \left|\langle l, v_n(t) - v_n(t_m)\rangle_{X^*,X}\right| \\
&+ \left|\langle l, u(t_m) - u(t)\rangle_{X^*,X}\right| \\
&\leq \varepsilon + 2\,\|l : X^*\|\, f(t_m - t) \\
&\leq (1 + 2\,\|l : X^*\|)\,\varepsilon.
\end{aligned}$$

Q.E.D.

The next lemma is important for our proof of continuity.

**Lemma 2.6.** *Let $(u_n, v_n)_{n\in\mathbb{Z}_{>0}}$ be a bounded $H^{1/2} \times H^{1/2}$ sequence which converges $(u, v) \in H^{1/2} \times H^{1/2}$ in $L^2 \times L^2$ topology. If $E(u_n, v_n)$ converges to $E(u, v)$, then $\|(u, v) - (u_n, v_n) : \mathscr{H}\| \to 0$.*

***proof***.

$$
\begin{aligned}
&\|(u_n,v_n)-(u,v):\mathscr{H}\|^2\\
&=2\mathrm{Re}\Big((u,v),(u,v)-(u_n,v_n)\Big)_{\mathscr{H}}-\|(u,v):\mathscr{H}\|^2+\|(u_n,v_n):\mathscr{H}\|^2\\
&=2\mathrm{Re}\Big((u,v),(u,v)-(u_n,v_n)\Big)_{\mathscr{H}}\\
&+E(u,v)-E(u_n,v_n)-\mathrm{Re}(\lambda v,u^2)+\mathrm{Re}(\lambda v_n,u_n^2). \qquad \text{(III.2.3)}
\end{aligned}
$$

The first coordinate of the first term on the right hand side is estimated as follows:

$$
\begin{aligned}
&\left|\left(\sqrt{m_u^2-\Delta}u,u-u_n\right)\right|\\
&\le\left|\left(\rho(\rho-\Delta)^{-1}\sqrt{m_u^2-\Delta}u,u-u_n\right)\right|+\left|\left(\Delta(\rho-\Delta)^{-1}\sqrt{m_u^2-\Delta}u,u-u_n\right)\right|\\
&\le\left\|\rho(\rho-\Delta)^{-1}\sqrt{m_u^2-\Delta}u:L^2\right\|\|u-u_n:L^2\|\\
&+\left\|\left(\Delta(\rho-\Delta)^{-1}\sqrt{m_u^2-\Delta}u:H^{-1/2}\right\|\|u-u_n:H^{1/2}\|
\end{aligned}
$$

Then, the left hand side is arbitrarily small for sufficiently large $\rho$ and $n$. By Hölder and Gagliardo-Nirenberg inequalities,

$$
\begin{aligned}
&\left|(v,u^2)-(v_n,u_n^2)\right|\\
&\le\left\|v-v_n:L^2\right\|\left\|u:L^4\right\|^2+\left\|v_n:L^2\right\|\left\|u+u_n:L^4\right\|\left\|u-u_n:L^4\right\|\\
&\lesssim\|u:H^{1/2}\|^2\left\|v-v_n:L^2\right\|+\\
&+\|v_n:L^2\|(\|u:H^{1/2}\|+\|u_n:H^{1/2}\|)^{3/2}\left\|u-u_n:L^2\right\|^{1/2}.
\end{aligned}
$$

Then, the remain terms of (III.2.3) go to 0 as $n$ increases. Q.E.D.

# 3 Proof of Theorem 3

In this section, we separate the proof into two parts, the proof for the construction of solutions, and the proof for the continuous dependence on initial values.

## 3.1 Construction of Solutions

Here, we construct solutions for $H^{1/2}\times H^{1/2}$ initial data. The finiteness of charge and energy plays an important role for our construction. Let $(u_0,v_0)\in H^{1/2}\times H^{1/2}$ and $(u_{0,n},v_{0,n})_{n\in\mathbb{Z}_0}\subset H^1\times H^1$ satisfy $(u_n,v_n)\to(u,v)$ in $H^{1/2}\times H^{1/2}$ and

$$
\left\|(u_{0,n},v_{0,n}):H^{1/2}\times H^{1/2}\right\|\lesssim\left\|(u_0,v_0):H^{1/2}\times H^{1/2}\right\|.
$$

Let $(u_n, v_n) \in C(\mathbb{R}, H^1 \times H^1)$ be the solutions of (III.2.1) for the initial data $(u_{0,n}, v_{0,n})$. For $t_0 < t_1$,

$$
\begin{aligned}
&\left\|u_n(t_1) - u_n(t_0) : L^2\right\| \\
&\leq \left\|\{U_{m_u}(t_1 - t_0) - 1\}u_n(t_0) : L^2\right\| + \int_{t_0}^{t_1} \left\|\overline{u_n}v_n(t') : L^2\right\| dt' \\
&\leq \left\|\left(2 \wedge (t_1 - t_0)\sqrt{m_u^2 - \Delta}\right)u_n(t_0) : L^2\right\| + \int_{t_0}^{t_1} \left\|u_n(t') : H^{1/2}\right\| \left\|v_n(t') : H^{1/2}\right\| dt' \\
&\lesssim \left\|u_0 : H^{1/2}\right\| |t_1 - t_0|^{1/2} + M(u_0, v_0)^2 |t_1 - t_0|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left\|v_n(t_1) - v_n(t_0) : L^2\right\| \\
&\leq \left\|\{U_{m_v}(t_1 - t_0) - 1\}v_n(t_0) : L^2\right\| + \int_{t_0}^{t_1} \left\|u_n(t')^2 : L^2\right\| dt' \\
&\leq \left\|\left(2 \wedge (t_1 - t_0)\sqrt{m_v^2 - \Delta}\right)v_n(t_0) : L^2\right\| + \int_{t_0}^{t_1} \left\|u_n(t') : H^{1/2}\right\|^2 dt' \\
&\lesssim \left\|v_0 : H^{1/2}\right\| |t_1 - t_0|^{1/2} + M(u_0, v_0)^2 |t_1 - t_0|.
\end{aligned}
$$

Then, by Lemma 2.5, we have $(u, v) \in C(\mathbb{R}, L^2 \times L^2) \cap L^\infty(\mathbb{R}, H^{1/2} \times H^{1/2})$ such that a subsequence $(\tilde{u}_n, \tilde{v}_n)_{n \in \mathbb{Z}_{>0}}$ of $(u_n, v_n)_{n \in \mathbb{Z}_{>0}}$ converges pointwisely weakly in $H^{1/2} \times H^{1/2}$ and

$$
\begin{aligned}
&\left\|(u, v)(t_1) - (u, v)(t_0) : L^2 \times L^2\right\| \\
&\lesssim \left\|(u_0, v_0) : L^2\right\| |t_1 - t_0|^{1/2} + M(u_0, v_0)^2 |t_1 - t_0|.
\end{aligned}
$$

In addition, by (NSR2) ,

$$
\|(\partial_t u_n, \partial_t v_n)(t) : H^{-1/2} \times H^{-1/2}\| \lesssim M(u_0, v_0) + M(u_0, v_0)^2,
$$

and

$$
\begin{aligned}
&\|\partial_t u_n(t_1) - \partial_t u_n(t_0) : H^{-1}\| \\
&= \left\|\sqrt{{m_u}^2 - \Delta}\, u_n(t_1) - \sqrt{{m_u}^2 - \Delta}\, u_n(t_0) : H^{-1}\right\| \\
&+ \left\|\overline{u_n}v_n(t_1) - \overline{u_n}v_n(t_0) : H^{-1}\right\| \\
&\lesssim (1 + M(u_0, v_0)))\|(u_n, v_n)(t_1) - (u_n, v_n)(t_0) : L^2\| \\
&\lesssim (1 + M(u_0, v_0)) \left(\left\|(u_0, v_0) : L^2 \times L^2\right\| |t_1 - t_0|^{1/2} + M(u_0, v_0)|t_1 - t_0|\right).
\end{aligned}
$$

Then, by Lemma 2.5, we have a subsequence of $(\tilde{u}_n, \tilde{v}_n)_{n \in \mathbb{Z}_{>0}}$, still denoted by $(\tilde{u}_n, \tilde{v}_n)_{n \in \mathbb{Z}_{>0}}$, and $(u', v') \in C(\mathbb{R}, H^{-1} \times H^{-1}) \cap L^\infty(\mathbb{R}, H^{-1/2} \times H^{-1/2})$ such that $(\partial_t \tilde{u}_n, \partial_t \tilde{v}_n)_{n \in \mathbb{Z}_{>0}}$ converges to

$(u', v')$ pointwisely weakly in $H^{-1/2} \times H^{-1/2}$. In addition, let $l \in H^{1/2}$ and let $\rho > 0$. Then,

$$\begin{aligned}
&\left| \left\langle l, \tilde{u}_n(t) - u_0 - \int_0^t u'(t')dt' \right\rangle_{L^2,L^2} \right| \\
&\leq \|l : L^2\| \|u_{0,n} - u_0 : L^2\| + \int_0^t \left| \langle l, \partial_t \tilde{u}_n(t') - u'(t') \rangle_{H^{1/2},H^{-1/2}} \right| dt' \\
&\leq \|l : L^2\| \|u_{0,n} - u_0 : L^2\| + \int_0^t \left| \left\langle \rho(\rho-\Delta)^{-1} l, \partial_t \tilde{u}_n(t') - u'(t') \right\rangle_{H^1,H^{-1}} \right| dt' \\
&+ t \left\| \partial_t \tilde{u}_n - u' : L^\infty([0,t], H^{-1/2}) \right\| \left\| \Delta(\rho-\Delta)^{-1} l : H^{1/2} \right\|.
\end{aligned}$$

Since $(\partial_t \tilde{u}_n)_{n\in\mathbb{Z}_{>0}}$ converges weakly uniformly on $[0,t]$, the right hand side goes to 0 as $\lambda$ and $n$ increase. This shows

$$u(t) = u_0 + \int_0^t u'(t)dt', \quad \partial_t u = u', \quad \text{in} \quad H^{-1/2}.$$

We also have $\partial_t v = v'$. Then, by (NSR2) , we have $(F, G) \in C(\mathbb{R}, H^{-1} \times H^{-1}) \cap L^\infty(\mathbb{R}, H^{-1/2} \times H^{-1/2})$ to which $(\overline{\tilde{u}_n}\tilde{v}_n(t), \tilde{u}_n^2(t))_{n\in\mathbb{Z}_{>0}}$ converges weakly in $H^{-1/2} \times H^{-1/2}$. What is left is to show $(\overline{u}v, u^2) = (F, G)$. Let $\psi \in C^\infty(\mathbb{R}, [0,1])$ with $\chi_{[-1,1]} \leq \psi \leq \chi_{[-2,2]}$ and $t \in \mathbb{R}$, where $\chi$ is a characteristic function. By a bilinear estimate for Sobolev norms,

$$\begin{aligned}
\left\| (\psi u_n(t), \psi v_n)(t) : H^{1/4} \times H^{1/4} \right\| &\lesssim \left\| \psi : H^{1/2} \right\| \left\| (u_n, v_n)(t) : H^{1/2} \times H^{1/2} \right\| \\
&\lesssim \left\| \psi : H^{1/2} \right\| M(u_0, v_0).
\end{aligned}$$

Then, by the Rellich's theorem, there is a subsequence of $((\tilde{u}_n(t), \tilde{v}_n(t))_{n\in\mathbb{Z}_{>0}}$, still denoted by $(\tilde{u}_n(t), \tilde{v}_n(t))_{n\in\mathbb{Z}_{>0}}$, such that $((\psi\tilde{u}_n, \psi\tilde{v}_n)(t))_{n\in\mathbb{Z}_{>0}}$ converges in $L^2 \times L^2$. Let $((u_\psi(t), v_\psi(t))_{n\in\mathbb{Z}_{>0}}$ be the limit of $(\psi\tilde{u}_n(t), \psi\tilde{v}_n(t))_{n\in\mathbb{Z}_{>0}}$. Then, for any $l \in L^2$,

$$\begin{aligned}
&\left| \langle l, u_\psi(t) - \psi u(t) \rangle_{L^2,L^2} \right| \\
&\leq \left| \langle l, u_\psi(t) - \psi\tilde{u}_n(t) \rangle_{L^2,L^2} \right| + \left| \langle \psi(-\cdot) l, u(t) - \tilde{u}_n(t) \rangle_{L^2,L^2} \right| \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

This means $(u_\psi, v_\psi)(t) = (\psi u, \psi v)(t)$ and $(\tilde{u}_n(t), \tilde{v}_n(t)) \to (u(t), v(t))$ a.e. on $[-1,1]$. In addition,

$$\left\| (\psi\overline{u_n}v_n(t), \psi u_n'^2(t)) : H^{1/4} \times H^{1/4} \right\| \lesssim \left\| \psi : H^{1/2} \right\| M(u_0, v_0)^2.$$

Then, we have a subsequence of $((\tilde{u}_n(t), \tilde{v}_n(t))_{n\in\mathbb{Z}_{>0}}$, still denoted by the same symbol, such that $(\psi\overline{\tilde{u}_n}\tilde{v}_n(t), \psi\tilde{u}_n^2(t))_{n\in\mathbb{Z}_{>0}}$ converges in $L^2 \times L^2$. Let $(F_\psi(t), G_\psi(t))$ is the limit of $(\psi\overline{\tilde{u}_n}\tilde{v}_n(t), \psi\tilde{u}_n^2(t))_{n\in\mathbb{Z}_{>0}}$. For any $l \in H^{1/2}$,

$$\begin{aligned}
&\left| \langle l, F_\psi(t) - \psi F(t) \rangle_{H^{1/2},H^{-1/2}} \right| \\
&\leq \left| \left\langle \psi(-\cdot) l, F(t) - \overline{\tilde{u}_n}\tilde{v}_n(t) \right\rangle_{H^{1/2},H^{-1/2}} \right| + \left| \left\langle l, F_\psi(t) - \psi\overline{\tilde{u}_n}\tilde{v}_n(t) \right\rangle_{L^2,L^2} \right| \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

This means $(\overline{\tilde{u}_n}\tilde{v}_n(t))_{n\in\mathbb{Z}_{>0}}$ converges to $F$ a.e. on $[-1,1]$. Then, we have $F(t)=\overline{u}v(t)$ on $[-1,1]$. By translation $\psi$, $F(t)=\overline{u(t)}v(t)$ and $G(t)=u(t)^2$ on $\mathbb{R}$ are shown.

### 3.2 Proof of the Continuous Dependence on Time Variable and Initial Values

Here, we show the continuous dependence of solutions in $H^{1/2}\times H^{1/2}$ on time and initial values. For the $H^{1/2}$ solutions which are the weak limit of a subsequence of smooth approximations, the continuous dependence of solutions on time in $L^2\times L^2$ topology is already shown. Then, the energy conservation law and Lemma 2.6 show the continuous dependence on time in $H^{1/2}\times H^{1/2}$ topology. Next, we show the continuous dependence on initial values. Let $(u_{0,n},v_{0,n})_n$ be a $H^{1/2}\times H^{1/2}$ sequence which converges $(u_0,v_0)\in H^{1/2}\times H^{1/2}$ in $H^{1/2}\times H^{1/2}$ topology. Then, $E(u_{0,n},v_{0,n})$ converges $E(u_0,v_0)$. By the energy conservation law and Lemma 2.6, it is enough to show the continuous dependence of solutions in $L^2\times L^2$ with respect to the initial data. Let $(u,v)$ be the solutions for the initial data $(u_0,v_0)$ and let $(u_n,v_n)$ be the solutions for the initial data $(u_{0_n},v_{0,n})$ for any $n$. Then,

$$
\begin{aligned}
\frac{d}{dt}\left\|u(t)-u_n(t):L^2\right\|^2 &= 2\mathrm{Re}\Big(u(t)-u_n(t),\ \partial_t u(t)-\partial_t u_n(t)\Big)_{L^2}\\
&= 2\mathrm{Re}\Big(u(t)-u_n(t),-i\Big(\overline{u}v(t)-\overline{u_n}v_n(t)\Big)\Big)_{L^2}.
\end{aligned}
$$

For any $p<2<\infty$ and $0<t<1$,

$$
\begin{aligned}
&\Big(u(t)-u_n(t),\overline{u}v(t)-\overline{u_n}v_n(t)\Big)_{L^2}\\
&\lesssim \int (|u(t)|\vee|v(t)|\vee|u_n(t)|\vee|v_n(t)|)^{1+2/p}|u(t)-u_n(t)|^{1/p'}\\
&\qquad\cdot(|u(t)-u_n(t)|^{1/p'}+|v(t)-v_n(t)|^{1/p'})dx\\
&\le \Big\{\left\||u(t)|^{1+2/p}:L^p\right\|+\left\||v(t)|^{1+2/p}:L^p\right\|\\
&\qquad+\left\||u_n(t)|^{1+2/p}:L^p\right\|+\left\||v_n(t)|^{1+2/p}:L^p\right\|\Big\}\\
&\cdot\Big(\left\|u(t)-u_n(t):L^2\right\|^{2/p'}+\left\|u(t)-u_n(t):L^2\right\|^{1/p'}\left\|v(t)-v_n(t):L^2\right\|^{1/p'}\Big)\\
&\le \Big\{\left\|u(t):L^{p+2}\right\|^{1+2/p}+\left\|v(t):L^{p+2}\right\|^{1+2/p}\\
&\qquad+\left\|u_n(t):L^{p+2}\right\|^{1+2/p}+\left\|v_n(t):L^{p+2}\right\|^{1+2/p}\Big\}\\
&\cdot\left\|(u,v)(t)-(u_n,v_n)(t):L^2\times L^2\right\|^{2/p'}\\
&\lesssim (M(u_0,v_0))p^{1/2+1/p}\left\|(u,v)(t)-(u_n,v_n)(t):L^2\times L^2\right\|^{2/p'}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{d}{dt}\left\|v(t)-v_n(t):L^2\right\|^2\\
&\lesssim M(u_0,v_0)p^{1/2+1/p}\left\|(u,v)(t)-(u_n,v_n)(t):L^2\times L^2\right\|^{2/p'}
\end{aligned}
$$

$p^{1/p}$ goes to 1 as $p$ increases. Since for any $t > 0$,

$$f'(t) \le C_0 p f^{1-1/p}(t) \Longrightarrow f(t) \le (C_0 t + f(0)^{1/p})^p,$$

where $C_0$ is independent of $t$. Then,

$$\begin{aligned}
\limsup_{n\to\infty} \Big\{ & \big\| u(t) - u_n(t) : L^\infty([0, 1/C_0), L^2) \big\|^2 \\
& + \big\| u(t) - u_n(t) : L^\infty([0, 1/C_0), L^2) \big\|^2 \Big\} \\
\le & (C_0 t)^p \longrightarrow 0 \qquad \text{as} \quad p \to \infty.
\end{aligned}$$

This shows the locally continuous dependence of solutions on time in $L^2 \times L^2$.

## References


[1] N. Bournaveas, "Local well-posedness for a nonlinear Dirac equation in spaces of almost critical dimension," Discrete Contin. Dyn. Syst., **20**(2008) 605–616.

[2] N. Bournaveas, T. Candy, and S. Machihara, "Local and global well posedness for the Chern-Simons-Dirac system in one dimension," Differential Integral Equations, **25**(2012), 699–718.

[3] J. P. Borgna and D. F. Rial, "Existence of ground states for a one-dimensional relativistic Schrödinger equation," J. Math. Phys., **53**(2012).

[4] Y. Cho and T. Ozawa, "On the semirelativistic Hartree-type equation," SIAM J. Math. Anal., **38**(2006), 1060–1074.

[5] P. D'Ancona, D. Foschi, and S. Selberg, "Null structure and almost optimal local regularity for the Dirac-Klein-Gordon system," J. Eur. Math. Soc., **9**(2007), 877–899.

[6] M. Escobedo and L. Vega, "A semilinear Dirac equation in $H^s(\mathbf{R}^3)$ for $s > 1$," SIAM J. Math. Anal., **28**(1997), 338–362.

[7] J. Fröhlich and E. Lenzmann, "Blowup for nonlinear wave equations describing boson stars," Comm. Pure Appl. Math., **60**(2007), 1691–1705.

[8] K. Fujiwara, S. Machihara, and T. Ozawa, "Well-posedness for the Cauchy problem of a system of semirelativistic equations," preprint.

[9] J. Ginibre, Y. Tsutsumi, and G. Velo, "On the Cauchy problem for the Zakharov system," J. Funct. Anal., **151**(1997), 384–436.

[10] N. Hayashi, C. Li, and T. Ozawa, "Small data scattering for a system of nonlinear Schrödinger equations," Differ. Equ. Appl., **3**(2011), 415–426.

[11] N. Hayashi, C. Li, and P. I. Naumkin, "On a system of nonlinear Schrödinger equations in 2d," Differential Integral Equations, **24**(2011), 417–434.

[12] N. Hayashi, T. Ozawa, and K. Tanaka, "On a system of nonlinear Schrödinger equations with quadratic interaction," Ann. I. H. Poincaré, **30**(2013), 661–690.

[13] G. Hoshino and T. Ozawa, "Analytic smoothing effect for a system of nonlinear Schrödinger equations," to appear in Differ. Equ. Appl., **5**(2013), 395–408.

[14] J. Krieger, E. Lenzmann, and P. Raphaël, "Nondispersive solutions to the $L^2$-critical half-wave equation," Arch. Ration. Mech. Anal., **209**(2013), 61–129.

[15] E. Lenzmann, "Well-posedness for semi-relaivistic Hartree equations of critical type," Math. Phys. Anal. Geom., **10**(2007), 43–64.

[16] S. Machihara, M. Nakamura, K. Nakanishi, and T. Ozawa, "Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation," J. Funct. Anal., **219**(2005), 1–20.

[17] S. Machihara, M. Nakamura, and T. Ozawa, "Small global solutions for nonlinear Dirac equations," Differential Integral Equations, **17**(2004) 623–636.

[18] S. Machihara, K. Nakanishi, and T. Ozawa, "Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation," Rev. Mat. Iberoamericana, **19**(2003), 179–194.

[19] S. Machihara, K. Nakanishi, and K. Tsugawa, "Well-posedness for nonlinear Dirac equations in one dimension," Kyoto J. Math., **50**(2010), 403–451.

[20] T. Ogawa and T. Ozawa, "Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrödinger mixed problem," J. Math. Anal. Appl., **155**(1991), 531–540.

[21] T. Ogawa, "A proof of Trudinger's inequality and its application to nonlinear Schrödinger equations," Nonlinear Anal., **14**(1990) 765–769.

[22] T. Ozawa, "Remarks on proofs of conservation laws for nonlinear Schrödinger equations," Calc. Var. Partial Differential Equations, v**25**(2006), 403–408.

[23] S. Selberg and A. Tesfahun, "Low regularity well-posedness of the Dirac-Klein-Gordon equations in one space dimension," Commun. Contemp. Math., **10**(2008), 181–194.

[24] M. V. Vladimirov, "On the solvability of a mixed problem for a nonlinear equation of Schrödinger type," Dokl. Akad. Nauk SSSR, **275**(1984), 780–783.

# Part IV

# Appendix A: Exact remainder formula for the Young inequality and applications

International Journal of Mathematical Analysis, Vol. 7, 2013, no. 55, 2723 - 2735
**HIKARI Ltd, www.m-hikari.com**
**http://dx.doi.org/10.12988/**



**Abstract**

We present explicit formulae for the remainder arising in the Young, Hölder, and Clarkson inequalities.

**Mathematics Subject Classification:** 26D15

**Keywords:** Young's inequality, Hölder's inequality.

## 1 Introduction

In this part we show the way to recognize the well-known inequalities of Young, Hölder, and Clarkson inequalities as special cases of the identities to be presented below, where necessary and sufficient conditions for those inequalities to be equalities are given in an explicit way. This work is motivated by [7], where explicit remainder terms of those inequalities are given. Here we give a more precise description on the remainder with a direct and straightforward calculation. Moreover, we improve available inequalities on the remainder. In addition, in the case where the remainder is expressed as polynomials, we give an exact formula with explicit coefficients.

Throughout this part, the following remainder function [7] plays an important role :

$$R(\theta; a, b) = \theta a + (1-\theta)b - a^\theta b^{1-\theta}, \tag{IV.1.1}$$

where $a, b > 0$ and $0 < \theta < 1$.

The standard Young inequality is described as

$$R(\theta; a, b) \geq 0 \tag{IV.1.2}$$

for any $\theta \in (0,1)$, $a, b > 0$. Then, the standard Hölder inequality follows from the identity

$$\int_\Omega |fg| d\mu = \|f\|_p \|g\|_{p'} \left(1 - \int_\Omega R\left(\frac{1}{p}; \frac{|f|^p}{\|f\|_p^p}, \frac{|g|_{p'}^{p'}}{\|g\|_{p'}^{p'}}\right) d\mu\right) \tag{IV.1.3}$$

for any $f \in L^p(\Omega, \mu)\backslash\{0\}$ and $g \in L^{p'}(\Omega, \mu)\backslash\{0\}$, where $L^q(\Omega, \mu)$ is the Banach space of $q$-th integrable functions on a measure space $(\Omega, \mu)$ with norm $\|\cdot\|_q$, $q \in (1, \infty)$, and $p'$ is the dual exponent to $p$ defined by $1/p + 1/p' = 1$.

The purpose in this part is to give a clear understanding of the standard Young and Hölder inequalities by an explicit description of the remainder function (IV.1.1). In Section 2, we prepare basic identities to be used for the main theorems. In Section 3, we present explicit formulae for the remainder function and their application. In Section 4, we study the remainder terms for the Clarkson inequality.

There are many papers on the related subjects. We refer the reader to [1–3, 3, 4, 4–6, 8, 8, 11] and references therein.

## 2 Basic Identities

The following elementary identities are basic in this part.

**Proposition 2.1.** *Let $p > 0$. Then, the following equalities*

$$x^p - y^p - px^{p-1}(x-y) = (1-p)p\int_0^1 t(tx+(1-t)y)^{p-2}dt(x-y)^2, \tag{IV.2.1}$$

$$x^p - y^p - py^{p-1}(x-y) = p(p-1)\int_0^1 t(ty+(1-t)x)^{p-2}dt(x-y)^2 \tag{IV.2.2}$$

*hold for all $x, y > 0$.*

***proof.*** For completeness we give a proof, though it is elementary. By a simple and straightforward calculation, we have

$$\begin{aligned}
&x^p - y^p - px^{p-1}(x-y) = (1-p)(x^p-y^p) + p(x^{p-1}-y^{p-1})y \\
&= (1-p)p\int_0^1 (tx+(1-t)y)^{p-1}dt(x-y) \\
&+ p(p-1)\int_0^1 (tx+(1-t)y)^{p-2}dt(x-y)y \\
&= (1-p)p\int_0^1 t(tx+(1-t)y)^{p-2}dt(x-y)^2, \\
&x^p - y^p - py^{p-1}(x-y) = px(x^{p-1}-y^{p-1}) - (p-1)(x^p-y^p) \\
&= p(p-1)x\int_0^1 (ty+(1-t)x)^{p-2}dt(x-y) \\
&- p(p-1)\int_0^1 (ty+(1-t)x)^{p-1}dt(x-y) \\
&= p(p-1)\int_0^1 t(ty+(1-t)x)^{p-2}dt(x-y)^2,
\end{aligned}$$

as required. Q.E.D.

## 3 Explicit Formulae for the Remainder Function

In this section we present explicit formulae for the remainder function and their applications.

**Theorem 1.** *Let $\theta \in (0,1)$. Then, the following equalities*

$$\begin{aligned}
&R(\theta; a, b) \\
&= \theta(1-\theta)\Bigg[\theta a^{\theta}\int_0^1 t(ta+(1-t)b)^{-\theta-1}dt \\
&\qquad + (1-\theta)b^{1-\theta}\int_0^1 t(tb+(1-t)a)^{\theta-2}dt\Bigg](a-b)^2 \qquad \text{(IV.3.1)}\\
&= \theta(1-\theta)\left[\int_0^1\int_0^t (ta+(1-t)b)^{\theta-1}(sa+(1-s)b)^{-\theta}dsdt\right](a-b)^2 \qquad \text{(IV.3.2)}
\end{aligned}$$

*hold for all $a, b > 0$.*

**Corollary 3.1.** *Let $\theta \in (0,1)$. Then, the following inequalities*

$$\begin{aligned}
&\frac{\theta(1-\theta)}{2}\frac{(a-b)^2}{\min(a,b)} \geq R(\theta; a, b) \\
&\qquad \geq \frac{\theta(1-\theta)}{2}\frac{(a-b)^2}{\max(a,b)} \geq \frac{\theta(1-\theta)}{2}\min(a,b)\left(\log\frac{a}{b}\right)^2 \qquad \text{(IV.3.3)}
\end{aligned}$$

*hold for all $a, b > 0$.*

**Remark 3.2.** *In [7], it is shown that*

$$\begin{aligned}
R(\theta; a, b) &= \left(\log\frac{a}{b}\right)^2\int_0^1 \min(\theta(1-t), (1-\theta)t)a^t b^{1-t}dt \\
&\geq \frac{\theta(1-\theta)}{2}\min(a,b)\left(\log\frac{a}{b}\right)^2
\end{aligned}$$

*for all $a, b > 0$. Therefore,* (IV.3.3) *is an improvement of the last inequality.*

**Remark 3.3.** *The first and second inequalities in* (IV.3.3) *are optimal in the sense that*

$$\begin{aligned}
&\lim_{\delta\downarrow 0}\sup_{0<|a-b|<\delta}\max(a,b)(a-b)^{-2}R(\theta; a, b) \\
&= \lim_{\delta\downarrow 0}\inf_{0<|a-b|<\delta}\min(a,b)(a-b)^{-2}R(\theta; a, b) = \frac{\theta(1-\theta)}{2}.
\end{aligned}$$

**Proof of Theorem 1.** For the first equality, we rewrite the remainder function as

$$\begin{aligned}
&R(\theta; a, b) \\
&= \theta a^{\theta}(a^{1-\theta} - b^{1-\theta} - (1-\theta)a^{-\theta}(a-b)) + (1-\theta)(b^{\theta} - a^{\theta} - \theta b^{\theta-1}(b-a))b^{1-\theta} \\
&= \theta^2(1-\theta)a^{\theta}\int_0^1 t(ta+(1-t)b)^{-\theta-1}dt(a-b)^2 \\
&+ (1-\theta)^2\theta b^{1-\theta}\int_0^1 t(tb+(1-t)a)^{\theta-2}dt(a-b)^2,
\end{aligned}$$

where we have used (IV.2.1) with $(x, y, p) = (a, b, 1-\theta)$ and $(x, y, p) = (b, a, \theta)$. For the second equality, we rewrite the remainder function as

$$
\begin{aligned}
&R(\theta; a, b) = \theta(1-\theta)a^{\theta}\int_b^a \xi^{-\theta}d\xi - \theta(1-\theta)b^{1-\theta}\int_b^a \xi^{\theta-1}d\xi \\
&= \theta(1-\theta)\int_b^a \xi^{-\theta}(a^{\theta}-\xi^{\theta})d\xi + \theta(1-\theta)\int_b^a \xi^{\theta-1}(\xi^{1-\theta}-b^{1-\theta})d\xi \\
&= \theta^2(1-\theta)\int_b^a \xi^{-\theta}\left(\int_\xi^a \eta^{\theta-1}d\eta\right)d\xi + \theta(1-\theta)^2\int_b^a \xi^{\theta-1}\left(\int_b^\xi \eta^{-\theta}d\eta\right)d\xi \\
&= \theta(1-\theta)\int_b^a \left(\int_b^\xi \eta^{-\theta}d\eta\right)\xi^{\theta-1}d\xi \\
&= \theta(1-\theta)\int_b^a \left(\int_0^1 (s\xi+(1-s)b)^{-\theta}ds(\xi-b)\right)\xi^{\theta-1}d\xi \\
&= \theta(1-\theta)\int_0^1 (ta+(1-t)b)^{\theta-1} \\
&\quad \cdot \int_0^1 \left(s\Big(ta+(1-t)b\Big)+(1-s)b\right)^{-\theta}ds\Big(\Big(ta+(1-t)b\Big)-b\Big)dt(a-b) \\
&= \theta(1-\theta)\left[\int_0^1\int_0^t (ta+(1-t)b)^{\theta-1}(sa+(1-s)b)^{-\theta}ds\ dt\right](a-b)^2.
\end{aligned}
$$

Q.E.D.

**Proof of Corollary 3.1.** The first and second inequalities in (IV.3.3) follow from the inequalities

$$
\frac{1}{\min(a,b)} \geq 2\int_0^1\int_0^t (ta+(1-t)b)^{\theta-1}(sa+(1-s)b)^{-\theta}ds\ dt \geq \frac{1}{\max(a,b)}.
$$

By substituting $x = a/b$ or $x = b/a$, the last inequality in (IV.3.3) follows from the inequality

$$
(x-1)^2 \geq x(\log x)^2
$$

for all $x > 0$. Q.E.D.

We give another explicit formula for the remainder function, which implies a polynomial representation for rational $\theta$.

**Theorem 2.** *Let $m, n$ be two positive real numbers. Then, the following equality*

$$
\begin{aligned}
&R\left(\frac{m}{m+n}; x^{m+n}, y^{m+n}\right) \\
&= \frac{mn}{m+n}\left[(n-1)\int_0^1 t(ty+(1-t)x)^{n-2}dt\ x^m \right. \\
&\qquad \left. +(m-1)\int_0^1 t(tx+(1-t)y)^{m-2}dt\ y^n + x^{m-1}y^{n-1}\right](x-y)^2 \qquad \text{(IV.3.4)}
\end{aligned}
$$

*holds for all $x, y > 0$.*

**Corollary 3.4.** *Let $m, n$ be two positive integers. Then, the following equality*

$$\begin{aligned}&\frac{m}{m+n}x^{m+n}+\frac{n}{m+n}y^{m+n}-x^my^n\\&=\frac{(x-y)^2}{m+n}\left[m\sum_{k=1}^{n-1}kx^{m+n-1-k}y^{k-1}+n\sum_{k=1}^{m-1}ky^{m+n-1-k}x^{k-1}+mnx^{m-1}y^{n-1}\right]\end{aligned}\tag{IV.3.5}$$

*holds as polynomials in $x, y$.*

**Remark 3.5.** *Identity* (IV.3.5) *was first proved in [7] by an induction argument. See also [3] for a characterization of the Young inequality in the framework of polynomials.*

**Proof of Theorem 2.** We rewrite the remainder function as

$$\begin{aligned}&R\left(\frac{m}{m+n};x^{m+n},y^{m+n}\right)\\&=\frac{m}{m+n}x^m(x^n-y^n-ny^{n-1}(x-y))+\frac{n}{m+n}(y^m-x^m-mx^{m-1}(y-x))y^n\\&\quad+\frac{mn}{m+n}x^{m-1}y^{n-1}(x-y)^2\end{aligned}$$

and apply (IV.2.1) and (IV.2.2) to obtain (IV.3.4). Q.E.D.

**Proof of Corollary 3.4.** We complete the integral on the right hand side of (IV.3.4). For instance,

$$\begin{aligned}&(n-1)\int_0^1 t(ty+(1-t)x)^{n-2}dt\\&=(n-1)\sum_{k=0}^{n-2}\binom{n-2}{k}x^{n-2-k}y^k\int_0^1 t^{k+1}(1-t)^{n-2-k}dt\\&=(n-1)\sum_{k=0}^{n-2}\frac{(n-2)!}{(n-2-k)!k!}\frac{(n-2-k)!(k+1)!}{n!}x^{n-2-k}y^k\\&=\frac{1}{n}\sum_{k=0}^{n-2}(k+1)x^{n-2-k}y^k.\end{aligned}$$

Q.E.D.

**Remark 3.6.** *Here are examples of* (IV.3.5) *up to degree 10.*

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 - xy = \frac{1}{2}(x-y)^2,$$
$$\frac{1}{3}x^3 + \frac{2}{3}y^3 - xy^2 = \frac{1}{3}\Big\{x + 2y\Big\}(x-y)^2,$$
$$\frac{1}{4}x^4 + \frac{3}{4}y^4 - xy^3 = \frac{1}{4}\Big\{x^2 + 2xy + 3y^2\Big\}(x-y)^2,$$
$$\frac{1}{5}x^5 + \frac{4}{5}y^5 - xy^4 = \frac{1}{5}\Big\{x^3 + 2x^2y + 3xy^2 + 4y^3\Big\}(x-y)^2,$$
$$\frac{1}{6}x^6 + \frac{5}{6}y^6 - xy^5 = \frac{1}{6}\Big\{x^4 + 2x^3y + 3x^2y^2 + 4xy^3 + 5y^4\Big\}(x-y)^2,$$
$$\frac{1}{7}x^7 + \frac{6}{7}y^7 - xy^6 = \frac{1}{7}\Big\{x^5 + 2x^4y + 3x^3y^2 + 4x^2y^3 + 5y^4 + 6y^5\Big\}(x-y)^2,$$
$$\frac{1}{8}x^8 + \frac{7}{8}y^8 - xy^7$$
$$= \frac{1}{8}\Big\{x^6 + 2x^5y + 3x^4y^2 + 4x^3y^3 + 5x^2y^4 + 6xy^5 + 7y^6\Big\}(x-y)^2,$$
$$\frac{1}{9}x^9 + \frac{8}{9}y^9 - xy^8$$
$$= \frac{1}{9}\Big\{x^7 + 2x^6y + 3x^5y^2 + 4x^4y^3 + 5x^3y^4 + 6x^2y^5 + 7xy^6 + 8y^7\Big\}(x-y)^2,$$
$$\frac{1}{10}x^{10} + \frac{9}{10}y^{10} - xy^9 = \frac{1}{10}\Big\{x^8 + 2x^7y + 3x^6y^2 + 4x^5y^3 + 5x^4y^4$$
$$+ 6x^3y^5 + 7x^2y^6 + 8xy^7 + 9y^8\Big\}(x-y)^2,$$

$$\frac{2}{4}x^4 + \frac{2}{4}y^4 - x^2y^2 = \frac{1}{4}\Big\{2x^2 + 4xy\Big\}(x-y)^2,$$
$$\frac{2}{5}x^5 + \frac{3}{5}y^5 - x^2y^3 = \frac{1}{5}\Big\{2x^3 + 4x^2y + 6xy^2 + 3y^3\Big\}(x-y)^2,$$
$$\frac{2}{6}x^6 + \frac{4}{6}y^6 - x^2y^4 = \frac{1}{6}\Big\{2x^4 + 4x^3y + 6x^2y^2 + 8xy^3 + 4y^4\Big\}(x-y)^2,$$
$$\frac{2}{7}x^7 + \frac{5}{7}y^7 - x^2y^5 = \frac{1}{7}\Big\{2x^5 + 4x^4y + 6x^3y^2 + 8x^2y^3 + 10xy^4 + 5y^5\Big\}(x-y)^2,$$
$$\frac{2}{8}x^8 + \frac{6}{8}y^8 - x^2y^6$$
$$= \frac{1}{8}\Big\{2x^6 + 4x^5y + 6x^4y^2 + 8x^3y^3 + 10x^2y^4 + 12xy^5 + 6y^6\Big\}(x-y)^2,$$
$$\frac{2}{9}x^9 + \frac{7}{9}y^9 - x^2y^7$$
$$= \frac{1}{9}\Big\{2x^7 + 4x^6y + 6x^5y^2 + 8x^4y^3 + 10x^3y^4 + 12x^2y^5 + 14xy^6 + 7y^7\Big\}(x-y)^2,$$

$$\frac{2}{10}x^{10} + \frac{8}{10}y^{10} - x^2y^8 = \frac{1}{10}\Big\{2x^8 + 4x^7y + 6x^6y^2 + 8x^5y^3 + 10x^4y^4 + 12x^3y^5 + 14x^2y^6 + 16xy^7 + 8y^8\Big\}(x-y)^2,$$

$$\frac{3}{6}x^6 + \frac{3}{6}y^6 - x^3y^3 = \frac{1}{6}\Big\{3x^4 + 6x^3y + 9x^2y^2 + 6xy^3\Big\}(x-y)^2,$$

$$\frac{3}{7}x^7 + \frac{4}{7}y^7 - x^3y^4 = \frac{1}{7}\Big\{3x^5 + 6x^4y + 9x^3y^2 + 12x^2y^3 + 8xy^4 + 4y^5\Big\}(x-y)^2,$$

$$\frac{3}{8}x^8 + \frac{5}{8}y^8 - x^3y^5 = \frac{1}{8}\Big\{3x^6 + 6x^5y + 9x^4y^2 + 12x^3y^3 + 15x^2y^4 + 10xy^5 + 5y^6\Big\}(x-y)^2,$$

$$\frac{3}{9}x^9 + \frac{6}{9}y^9 - x^3y^6 = \frac{1}{9}\Big\{3x^7 + 6x^6y + 9x^5y^2 + 12x^4y^3 + 15x^3y^4 + 18x^2y^5 + 12xy^6 + 6y^7\Big\}(x-y)^2,$$

$$\frac{3}{10}x^{10} + \frac{7}{10}y^{10} - x^3y^7 = \frac{1}{10}\Big\{3x^8 + 6x^7y + 9x^6y^2 + 12x^5y^3 + 15x^4y^4 + 18x^3y^5 + 21x^2y^6 + 14xy^7 + 7y^8\Big\}(x-y)^2,$$

$$\frac{4}{8}x^8 + \frac{4}{8}y^8 - x^4y^4 = \frac{1}{8}\Big\{4x^6 + 8x^5y + 12x^4y^2 + 16x^3y^3 + 12x^2y^4 + 8xy^5\Big\}(x-y)^2,$$

$$\frac{4}{9}x^9 + \frac{5}{9}y^9 - x^4y^5 = \frac{1}{9}\Big\{4x^7 + 8x^6y + 12x^5y^2 + 16x^4y^3 + 20x^3y^4 + 15x^2y^5 + 10xy^6 + 5y^7\Big\}(x-y)^2,$$

$$\frac{4}{10}x^{10} + \frac{6}{10}y^{10} - x^4y^6 = \frac{1}{10}\Big\{4x^8 + 8x^7y + 12x^6y^2 + 16x^5y^3 + 20x^4y^4 + 24x^3y^5 + 18x^2y^6 + 12xy^7 + 6y^8\Big\}(x-y)^2,$$

$$\frac{5}{10}x^{10} + \frac{5}{10}y^{10} - x^5y^5 = \frac{1}{10}\Big\{5x^8 + 10x^7y + 15x^6y^2 + 20x^5y^3 + 25x^4y^4 + 20x^3y^5 + 15x^2y^6 + 10xy^7\Big\}(x-y)^2.$$

## 4 Remainder Formulae for the Clarkson Inequality

In this section we study the remainder for the Clarkson inequality.

**Theorem 3.** *Let $p \geq 1$. Then, the following equalities*

$$\begin{aligned}
&2^{p-1}a^p + 2^{p-1}b^p - (a+b)^p \\
&= \frac{p(p-1)}{2}\left[\int_0^1 t(a+(1-t)a+tb)^{p-2}dt + \int_0^1 t(b+(1-t)b+ta)^{p-2}dt\right] \\
&\quad \cdot (a-b)^2 \\
&= \frac{p(p-1)}{2}\left[\int_0^1 \int_0^1 ((t+s)(a-b)+2b)^{p-2}dt\ ds\right](a-b)^2 \qquad \text{(IV.4.1)}
\end{aligned}$$

*hold for all $a, b > 0$.*

**Corollary 4.1.** *(1) Let $p \geq 2$. Then, the following inequalities*

$$2^{p-1}a^p + 2^{p-1}b^p - (a+b)^p \geq (2^{p-1}-1)|a-b|^p \geq |a-b|^p \qquad \text{(IV.4.2)}$$

*hold for all $a, b > 0$.*

*(2) Let $1 \leq p \leq 2$. Then, the following inequalities*

$$2^{p-1}a^p + 2^{p-1}b^p - (a+b)^p \leq (2^{p-1}-1)|a-b|^p \leq |a-b|^p \qquad \text{(IV.4.3)}$$

*hold for all $a, b > 0$.*

*(3) Let $p \geq 1$. Then, the following inequity*

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leq 2^{-\min(p-1,1)}(|a|^p + |b|^p) \qquad \text{(IV.4.4)}$$

*holds for all $a, b > 0$.*

**Proof of Theorem 3.** For the first equality, we write

$$\begin{aligned}
&2^{p-1}a^p + 2^{p-1}b^p - (a+b)^p \\
&= 2^{p-1}\left(a^p - \left(\frac{a+b}{2}\right)^p - p\left(\frac{a+b}{2}\right)^{p-1}\left(a - \frac{a+b}{2}\right)\right) \\
&\quad + 2^{p-1}\left(b^p - \left(\frac{a+b}{2}\right)^p - p\left(\frac{a+b}{2}\right)^{p-1}\left(b - \frac{a+b}{2}\right)\right) \\
&= 2^{p-1}p(p-1)\int_0^1 t\left(t\frac{a+b}{2} + (1-t)a\right)^{p-2} dt \left(\frac{a-b}{2}\right)^2 \\
&\quad + 2^{p-1}p(p-1)\int_0^1 t\left(t\frac{a+b}{2} + (1-t)b\right)^{p-2} dt \left(\frac{a-b}{2}\right)^2,
\end{aligned}$$

where we have used (IV.2.2) with $(x, y) = (a, (a+b)/2)$ and $(x, y) = (b, (a+b)/2)$.

For the second equality, we write

$$\begin{aligned}
&2^{p-1}a^p + 2^{p-1}b^p - (a+b)^p \\
&= \frac{p}{2}\int_b^a ((\xi + a)^{p-1} - (\xi + b)^{p-1})d\xi \\
&= \frac{p(p-1)}{2}\int_b^a \int_b^a (\xi + \eta)^{p-2} d\xi \ d\eta \\
&= \frac{p(p-1)}{2}\int_0^1 \int_0^1 ((t+s)a + (1-t-s)b + b)^{p-2} dt \ ds (a-b)^2.
\end{aligned}$$

Q.E.D.

**Proof of Corollary 4.1.** We may assume that $a \geq b > 0$.

(1) We estimate the right hand side of the last equality of (IV.4.1) from below as

$$\begin{aligned}
&\frac{p(p-1)}{2}\left[\int_0^1 \int_0^1 ((t+s)(a-b) + 2b)^{p-2} dt \ ds\right](a-b)^2 \\
&\geq \frac{p(p-1)}{2}(a-b)^p \int_0^1 \int_0^1 (t+s)^{p-2} dt \ ds = (2^{p-1} - 1)(a-b)^p
\end{aligned}$$

which yields (IV.4.2).

(2) We estimate the right hand side of the last equality of (IV.4.1) as

$$\begin{aligned}
&\frac{p(p-1)}{2}\left[\int_0^1 \int_0^1 ((t+s)(a-b) + 2b)^{p-2} dt \ ds\right](a-b)^2 \\
&\leq \frac{p(p-1)}{2}(a-b)^p \int_0^1 \int_0^1 (t+s)^{p-2} dt \ ds = (2^{p-1} - 1)(a-b)^p
\end{aligned}$$

which yields (IV.4.3).

(3) If $p \geq 2$, then (IV.4.4) follows directly from (IV.4.2). If $1 \leq p \leq 2$, then we apply (IV.4.3) to $\alpha = (a+b)/2$ and $\beta = (a-b)/2$ to obtain

$$\begin{aligned}
\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p &= 2^{1-p}\left(2^{p-1}|\alpha|^p + 2^{p-1}|\beta|^p\right) \\
&\leq 2^{1-p}\left((|\alpha| + |\beta|)^p + \Big||\alpha| - |\beta|\Big|^p\right) \\
&= 2^{1-p}\left(|\alpha + \beta|^p + |\alpha - \beta|^p\right) = 2^{1-p}(|a|^p + |b|^p),
\end{aligned}$$

which is precisely (IV.4.4).

Q.E.D.

**Remark 4.2.** *Inequalities* (IV.4.2) - (IV.4.4) *are optimal in the sense that they are equalities when $a = b$ for $p \geq 2$ and $b = 0$ for $p \leq 2$.*

**Theorem 4.** *Let $n$ be an integer with $n \geq 2$. Then, the following equality*

$$\begin{aligned} & 2^n x^n + 2^n y^n - 2(x+y)^n \\ & = (x-y)^2 \sum_{k=0}^{n-2} \left( (k+1) \sum_{j=k}^{n-2} \binom{n}{j+2} + (n-k-1) \sum_{j=n-k-2}^{n-2} \binom{n}{j+2} \right) \\ & \quad \cdot x^k y^{n-2-k} \end{aligned} \tag{IV.4.5}$$

*holds as polynomials in $x, y$.*

***proof***. We compute the integrals on the right hand side of the first equality of (IV.4.1). For instance, for an integer $m \geq 0$

$$\begin{aligned} & \int_0^1 t(y + (1-t)y + tx)^m dt \\ & = \sum_{j=0}^{m} \binom{m}{j} y^{m-j} \int_0^1 t((1-t)y + tx)^j dt \\ & = \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} y^{m-j+(j-k)} x^k \int_0^1 t^{k+1}(1-t)^{j-k} dt \\ & = \sum_{j=0}^{m} \sum_{k=0}^{j} \frac{m!}{(m-j)!j!} \frac{j!}{(j-k)!k!} \frac{(k+1)!(j-k)!}{(j+2)!} x^k y^{m-k} \\ & = \frac{1}{(m+2)(m+1)} \sum_{j=0}^{m} \sum_{k=0}^{j} (k+1) \binom{m+2}{j+2} x^k y^{m-k}. \end{aligned}$$

Q.E.D.

**Remark 4.3.** *Here are examples of* (IV.4.5) *up to degree 10.*

$$2^2a^2 + 2^2b^2 - 2(a+b)^2 = 2(a-b)^2,$$

$$2^3a^3 + 2^3b^3 - 2(a+b)^3 = 2\Big\{3b + 3a\Big\}(a-b)^2,$$

$$2^4a^4 + 2^4b^4 - 2(a+b)^4 = 2\Big\{7b^2 + 10ab + 7a^2\Big\}(a-b)^2,$$

$$2^5a^5 + 2^5b^5 - 2(a+b)^5 = 2\Big\{15b^3 + 25ab^2 + 25a^2b + 15a^3\Big\}(a-b)^2,$$

$$2^6a^6 + 2^6b^6 - 2(a+b)^6 = 2\Big\{31b^4 + 56ab^3 + 66a^2b^2 + 56a^3b + 31a^4\Big\}(a-b)^2,$$

$$\begin{aligned}&2^7a^7 + 2^7b^7 - 2(a+b)^7\\ &= 2\Big\{63b^5 + 119ab^4 + 154a^2b^3 + 154a^3b^2 + 119a^4b + 63a^5\Big\}(a-b)^2,\end{aligned}$$

$$\begin{aligned}2^8a^8 + 2^8b^8 - 2(a+b)^8 = 2\Big\{&127b^6 + 246ab^5 + 337a^2b^4\\ &+ 372a^3b^3 + 337a^4b^2 + 246a^5b + 127a^6\Big\}(a-b)^2,\end{aligned}$$

$$\begin{aligned}2^9a^9 + 2^9b^9 - 2(a+b)^9 = 2\Big\{&255b^7 + 501ab^6 + 711a^2b^5 + 837a^3b^4\\ &+ 837a^4b^3 + 711a^5b^2 + 501a^6b + 255a^7\Big\}(a-b)^2,\end{aligned}$$

$$\begin{aligned}&2^{10}a^{10} + 2^{10}b^{10} - 2(a+b)^{10}\\ &= 2\Big\{511b^8 + 1012ab^7 + 1468a^2b^6 + 1804a^3b^5 + 1930a^4b^4\\ &\qquad + 1804a^5b^3 + 1468a^6b^2 + 1012a^7b + 511a^8\Big\}(a-b)^2.\end{aligned}$$

## References


[1] J.M. Aldaz, "A stability version of Hölder's inequality," J. Math. Anal. Appl., **343**(2008), 842–852.

[2] J.A. Clarkson, "Uniformly convex spaces," Trans. Amer. Math. Soc., **40**(1936), 396–414.

[3] Z. Cvetkovski, "Inequalities," Springer, 2012.

[4] K.O. Friedrichs, "On Clarkson's inequalities," Comm. Pure Appl. Math., **23**(1970), 603–607.

[5] S. Furuichi and N. Minculete, "Alternative reverse inequalities for Young's inequality," J. Math. Inequal., **5**(2011), 595–600.

[6] K. Fujiwara and T. Ozawa, "Identities for the difference between the arithmetic and geometric means," preprint.

[7] X. Gao, M. Gao, and X. Shang, "A refinement of Hölder's inequality and applications," JIPAM. J. Inequal. Pure Appl. Math., **8**(2007), Article 44, 9.

[8] G.H. Hardy, J.E. Littlewood, and G. Pólya, "Inequalities," 2d. Ed., Cambridge, 1952.

[9] X.-L. Hu "An extension of Young's inequality and its application," Appl. Math. Comput., **219**(2013), 6393–6399.

[10] S. Kichenassamy, "Improving Hölder's inequality," Houston J. Math., **36**,(2010), 303–312.

[11] C. Morawetz, "Two $L^p$ inequalities," Bull. Amer. Math. Soc., **75**(1969), 1299–1302.

[12] J. Pečarić and V. Šimić, "A note on the Hölder inequality," JIPAM. J. Inequal. Pure Appl. Math., **7**(2006), Article 176, 3pp.


**Received: Month 12, 2013**

**Part V**

# Appendix B: Identities for the difference between the arithmetic and geometric means

**Abstract**

We prove a formula which expresses the difference between the arithmetic mean of variables of odd numbers and the corresponding geometric mean in the form of a linear combination of independent variables with coefficients given by sums of squares of polynomials.

Keywords: Arithmetic mean, Geometric mean, Hurwitz' identity, Muirhead's identity

MSC(2010): 26C, 26D05, 26D15.

## 1 Introduction

In this part we study expressions for the difference between the arithmetic mean of $n$ variables and their geometric mean. Regarding the difference between the arithmetic and geometric means, explicit formulae are given by Hurwitz [15] and Muirhead [16]. Moreover, in [14, Chapter 2], the difference is written as a sum of squares if $n$ is even, namely, for $n$ even, there exist polynomials $\{P_i;\ i \in I_n\}$ such that

$$\frac{1}{n}\sum_{j=1}^{n} x_j^n - \prod_{j=1}^{n} x_j = \sum_{i\in I_n} (P_i(x_1,\cdots,x_n))^2 \tag{V.1.1}$$

and $P_i(x_1,\cdots,x_n) = 0$ for all $i \in I_n$ if and only if $x_1 = \cdots = x_n$.

The purpose of this part is to present a formula which expresses the difference in the form of a linear combination of $n$ variables with coefficients given by sums of squares if $n$ is odd, namely, for $n$ odd, there exist polynomials $\{P_{ij};\ 1 \le i \le n,\ j \in J_n\}$ such that

$$\frac{1}{n}\sum_{j=1}^{n} x_j^n - \prod_{j=1}^{n} x_j = \sum_{i=1}^{n} x_i \sum_{j\in J_n} (P_{ij}(x_1,\cdots,x_n))^2 \tag{V.1.2}$$

and $P_{ij}(x_1,\cdots,x_n) = 0$ for all $i,j$ if and only if $x_1 = \cdots = x_n$. Particularly, an explicit formula is described for (V.1.2) (See Theorem 4.2 in Section 4 below). The formula is reduced to the well-known factorization for $n = 3$. It may be new even when $n = 5$ (see Section 5 below).

There is a large literature on the arithmetic and geometric means. We refer [1–11, 13, 17–20] for related subjects. In [19], an explicit representation of the difference $(x_1 + \cdots + x_n)^n - n^n x_1 \cdots x_n$ is given as a sum of $p_{ij}(x_1,\cdots x_n)(x_i - x_j)^2$ over all $i,j$ with $1 \le i < j \le n$, where $p_{i,j}$ are homogeneous polynomials of degree $n-2$. Our problem is different from that of [19]. In [10], our problem is studied in an algebraic setting but the corresponding representation is not written down explicitly.

This part is organized as follows. In Section 2, we collect some basic notation and identities. In Section 3, we prove preliminary propositions. In Section 4, we give the main theorems. An explicit formula for (V.1.2) is derived form the corresponding formula for (V.1.1), which is written essentially in [14]. In Section 5, we discuss examples.

## 2 Preliminaries

In this section, we collect basic notation and identities. First, we fix some basic notation and definitions. $\mathbb{Z}_{\geq 0}$ [resp. $\mathbb{Z}_{>0}$] denotes the set of all nonnegative [resp. positive] integers. For any set $A$, $\chi_A$ denotes its characteristic function. For any $n \in \mathbb{Z}_{>0}$, $\mathfrak{S}_n$ denotes the symmetry group of degree $n$. In a field, sum [resp. product] of elements with indices over the empty set is understood to be 0 [resp. 1]. In a ring, $a \equiv b \pmod{m}$ denotes the congruence relation modulo $m$.

**Definition 2.1.** *For any $n \in \mathbb{Z}_{>0}$, $p_n : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is defined as*

$$p_n(j) \equiv 2^j \pmod{2n+1} \quad \text{and} \quad 0 \leq p_n(j) \leq 2n.$$

**Proposition 2.2.** *Let $n \in \mathbb{Z}_{>0}$. The mapping $p_n : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is characterized inductively by*

$$p_n(0) = 1,$$
$$p_n(j+1) = \begin{cases} 2p_n(j) & \text{if } \ p_n(j) \leq n, \\ 2p_n(j) - 2n - 1 & \text{if } \ p_n(j) \geq n+1, \end{cases}$$

*for $j \in \mathbb{Z}_{\geq 0}$.*

Proof is straightforward and omitted. □

**Definition 2.3.** *For any $n \in \mathbb{Z}_{>0}$, $\lambda(n) \in \mathbb{Z}_{>0}$ is defined as*

$$\lambda(n) = \min\{j \in \mathbb{Z}_{>0} \ ; \ 2^j \equiv 1 \pmod{2n+1}\}.$$

**Remark 2.4.** *Let $\varphi$ be the Euler function. The theorem of Euler implies that $2^{\varphi(2n+1)} \equiv 1 \pmod{2n+1}$ and therefore $\lambda(n) \leq \varphi(2n+1)$. For any $n \in \mathbb{Z}_{>0}$, $2n+1 \geq 3$ and therefore $\lambda(n) \geq 2$. Hence $2 \leq \lambda(n) \leq \varphi(2n+1)$.*

**Remark 2.5.** *As is easily guessed by the behavior of the Euler function, it is difficult to describe a formation of the behavior of $\lambda$ in a simple way. In special cases, however, we have simple identities such as*

$$\lambda(2^l - 1) = l + 1, \tag{V.2.1}$$
$$\lambda(2^{l-1}) = 2l \tag{V.2.2}$$

*for any $l \in \mathbb{Z}_{>0}$.*

**Remark 2.6.** *Some explicit values of $p_n(j)$ and $\lambda(n)$ are given in Table 2.1. Values of $j$ are indicated on the top row, and $n$ on the left column. The values of $\lambda(n)$ are indicated on the last column. Other components of Table 2.1 shows values of $p_n(j)$.*

| $n \setminus j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | $\lambda(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |
| 2 | 2 | 4 | 3 | 1 | 2 | 4 | 3 | 1 | 2 | 4 | 3 | 1 | 2 | 4 | 3 | 1 | 2 | 4 | 3 | 1 | 2 | 4 | 3 | 1 | 2 | 4 | 3 | 1 | 4 |
| 3 | 2 | 4 | 1 | 2 | 4 | 1 | 2 | 4 | 1 | 2 | 4 | 1 | 2 | 4 | 1 | 2 | 4 | 1 | 2 | 4 | 1 | 2 | 4 | 1 | 2 | 4 | 1 | 2 | 3 |
| 4 | 2 | 4 | 8 | 7 | 5 | 1 | 2 | 4 | 8 | 7 | 5 | 1 | 2 | 4 | 8 | 7 | 5 | 1 | 2 | 4 | 8 | 7 | 5 | 1 | 2 | 4 | 8 | 7 | 6 |
| 5 | 2 | 4 | 8 | 5 | 10 | 9 | 7 | 3 | 6 | 1 | 2 | 4 | 8 | 5 | 10 | 9 | 7 | 3 | 6 | 1 | 2 | 4 | 8 | 5 | 10 | 9 | 7 | 3 | 10 |
| 6 | 2 | 4 | 8 | 3 | 6 | 12 | 11 | 9 | 5 | 10 | 7 | 1 | 2 | 4 | 8 | 3 | 6 | 12 | 11 | 9 | 5 | 10 | 7 | 1 | 2 | 4 | 8 | 3 | 12 |
| 7 | 2 | 4 | 8 | 1 | 2 | 4 | 8 | 1 | 2 | 4 | 8 | 1 | 2 | 4 | 8 | 1 | 2 | 4 | 8 | 1 | 2 | 4 | 8 | 1 | 2 | 4 | 8 | 1 | 4 |
| 8 | 2 | 4 | 8 | 16 | 15 | 13 | 9 | 1 | 2 | 4 | 8 | 16 | 15 | 13 | 9 | 1 | 2 | 4 | 8 | 16 | 15 | 13 | 9 | 1 | 2 | 4 | 8 | 16 | 8 |
| 9 | 2 | 4 | 8 | 16 | 13 | 7 | 14 | 9 | 18 | 17 | 15 | 11 | 3 | 6 | 12 | 5 | 10 | 1 | 2 | 4 | 8 | 16 | 13 | 7 | 14 | 9 | 18 | 17 | 18 |
| 10 | 2 | 4 | 8 | 16 | 11 | 1 | 2 | 4 | 8 | 16 | 11 | 1 | 2 | 4 | 8 | 16 | 11 | 1 | 2 | 4 | 8 | 16 | 11 | 1 | 2 | 4 | 8 | 16 | 6 |
| 11 | 2 | 4 | 8 | 16 | 9 | 18 | 13 | 3 | 6 | 12 | 1 | 2 | 4 | 8 | 16 | 9 | 18 | 13 | 3 | 6 | 12 | 1 | 2 | 4 | 8 | 16 | 9 | 18 | 11 |
| 12 | 2 | 4 | 8 | 16 | 7 | 14 | 3 | 6 | 12 | 24 | 23 | 21 | 17 | 9 | 18 | 11 | 22 | 19 | 13 | 1 | 2 | 4 | 8 | 16 | 7 | 14 | 3 | 6 | 20 |
| 13 | 2 | 4 | 8 | 16 | 5 | 10 | 20 | 13 | 26 | 25 | 23 | 19 | 11 | 22 | 17 | 7 | 14 | 1 | 2 | 4 | 8 | 16 | 5 | 10 | 20 | 13 | 26 | 25 | 18 |
| 14 | 2 | 4 | 8 | 16 | 3 | 6 | 12 | 24 | 19 | 9 | 18 | 7 | 14 | 28 | 27 | 25 | 21 | 13 | 26 | 23 | 17 | 5 | 10 | 20 | 11 | 22 | 15 | 1 | 28 |
| 15 | 2 | 4 | 8 | 16 | 1 | 2 | 4 | 8 | 16 | 1 | 2 | 4 | 8 | 16 | 1 | 2 | 4 | 8 | 16 | 1 | 2 | 4 | 8 | 16 | 1 | 2 | 4 | 8 | 5 |
| 16 | 2 | 4 | 8 | 16 | 32 | 31 | 29 | 25 | 17 | 1 | 2 | 4 | 8 | 16 | 32 | 31 | 29 | 25 | 17 | 1 | 2 | 4 | 8 | 16 | 32 | 31 | 29 | 25 | 10 |
| 17 | 2 | 4 | 8 | 16 | 32 | 29 | 23 | 11 | 22 | 9 | 18 | 1 | 2 | 4 | 8 | 16 | 32 | 29 | 23 | 11 | 22 | 9 | 18 | 1 | 2 | 4 | 8 | 16 | 12 |
| 18 | 2 | 4 | 8 | 16 | 32 | 27 | 17 | 34 | 31 | 25 | 13 | 26 | 15 | 30 | 23 | 9 | 18 | 36 | 35 | 33 | 29 | 21 | 5 | 10 | 20 | 3 | 6 | 12 | 36 |
| 19 | 2 | 4 | 8 | 16 | 32 | 25 | 11 | 22 | 5 | 10 | 20 | 1 | 2 | 4 | 8 | 16 | 32 | 25 | 11 | 22 | 5 | 10 | 20 | 1 | 2 | 4 | 8 | 16 | 12 |
| 20 | 2 | 4 | 8 | 16 | 32 | 23 | 5 | 10 | 20 | 40 | 39 | 37 | 33 | 25 | 9 | 18 | 36 | 31 | 21 | 1 | 2 | 4 | 8 | 16 | 32 | 23 | 5 | 10 | 20 |
| 21 | 2 | 4 | 8 | 16 | 32 | 21 | 42 | 41 | 39 | 35 | 27 | 11 | 22 | 1 | 2 | 4 | 8 | 16 | 32 | 21 | 42 | 41 | 39 | 35 | 27 | 11 | 22 | 1 | 14 |
| 22 | 2 | 4 | 8 | 16 | 32 | 19 | 38 | 31 | 17 | 34 | 23 | 1 | 2 | 4 | 8 | 16 | 32 | 19 | 38 | 31 | 17 | 34 | 23 | 1 | 2 | 4 | 8 | 16 | 12 |
| 23 | 2 | 4 | 8 | 16 | 32 | 17 | 34 | 21 | 42 | 37 | 27 | 7 | 14 | 28 | 9 | 18 | 36 | 25 | 3 | 6 | 12 | 24 | 1 | 2 | 4 | 8 | 16 | 32 | 23 |
| 24 | 2 | 4 | 8 | 16 | 32 | 15 | 30 | 11 | 22 | 44 | 39 | 29 | 9 | 18 | 36 | 23 | 46 | 43 | 37 | 25 | 1 | 2 | 4 | 8 | 16 | 32 | 15 | 30 | 21 |
| 25 | 2 | 4 | 8 | 16 | 32 | 13 | 26 | 1 | 2 | 4 | 8 | 16 | 32 | 13 | 26 | 1 | 2 | 4 | 8 | 16 | 32 | 13 | 26 | 1 | 2 | 4 | 8 | 16 | 8 |
| 26 | 2 | 4 | 8 | 16 | 32 | 11 | 22 | 44 | 35 | 17 | 34 | 15 | 30 | 7 | 14 | 28 | 3 | 6 | 12 | 24 | 48 | 43 | 33 | 13 | 26 | 52 | 51 | 49 | 52 |
| 27 | 2 | 4 | 8 | 16 | 32 | 9 | 18 | 36 | 17 | 34 | 13 | 26 | 52 | 49 | 43 | 31 | 7 | 14 | 28 | 1 | 2 | 4 | 8 | 16 | 32 | 9 | 18 | 36 | 20 |

Table 1: table of $p_n(j)$ and $\lambda(n)$

**Definition 2.7.** $\chi_+ = \chi_{2\mathbb{Z}}$, $\chi_- = \chi_{2\mathbb{Z}+1}$. *Namely,* $\chi_+$ *[resp.* $\chi_-$*] takes* 1 *on even [resp. odd] integers and* 0 *on odd [resp. even] integers.*

**Definition 2.8.** *For any* $n \in \mathbb{Z}_{>0}$, $\sigma_n^{\pm}(j) : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ *are defined as*

$$\sigma_n^{\pm}(j) = \sum_{i=1}^{j} 2^{j-i} \chi_{\pm}(p_n(i)).$$

**Remark 2.9.** *The following identities hold:*

$$\chi_{\pm}(j) = \frac{1 \pm (-1)^j}{2}, \tag{V.2.3}$$

$$p_n(j+1) = 2p_n(j) - \chi_-(p_n(j+1))(2n+1), \tag{V.2.4}$$

$$\sigma_n^+(j) + \sigma_n^-(j) = 2^j - 1. \tag{V.2.5}$$

**Proposition 2.10.** *For any* $n \in \mathbb{Z}_{>0}$ *and* $j \in \mathbb{Z}_{\geq 0}$,

$$2^j - p_n(j) = \sigma_n^-(j)(2n+1). \tag{V.2.6}$$

***proof***. Let $n \in \mathbb{Z}_{>0}$. We prove (V.2.6) by induction on $j \in \mathbb{Z}_{\geq 0}$. For $j = 0, 1$, (V.2.6) is trivial, since $p_n(0) = 1$, $p_n(1) = 2$, $\sigma_n^-(0) = \sigma_n^-(1) = 0$. Given (V.2.6), we have

$$\begin{aligned} 2^{j+1} - p_n(j+1) &= 2(2^j - p_n(j)) + \chi_-(p_n(j+1))(2n+1) \\ &= (2\sigma_n^-(j) + \chi_-(p_n(j+1))(2n+1) = \sigma_n^-(j+1)(2n+1), \end{aligned}$$

which completes the induction argument. Q.E.D.

**Proposition 2.11.** *For any* $m, n \in \mathbb{Z}_{>0}$,

$$p_n(m\lambda(n)) = 1, \tag{V.2.7}$$

$$\sigma_n^-(m\lambda(n)) = \frac{2^{m\lambda(n)} - 1}{2n+1}, \tag{V.2.8}$$

$$\sigma_n^+(m\lambda(n)) = \frac{2n(2^{m\lambda(n)} - 1)}{2n+1}. \tag{V.2.9}$$

***proof***. By definition, there exists a unique $q(n) \in \mathbb{Z}_{>0}$ such that $2^{\lambda(n)} - 1 = q(n)(2n+1)$, which together with the identity $2^{m\lambda(n)} - 1 = (2^{\lambda(n)} - 1) \sum_{j=0}^{m-1} 2^{\lambda(n)j}$ yields (V.2.7). By (V.2.6) and (V.2.7), we have (V.2.8). By (V.2.5) and (V.2.8), we have (V.2.9). Q.E.D.

## 3 Key Propositions

In this section, we prepare key propositions for the proof of the main theorems in the next section.

**Proposition 3.1.** *For any* $m, n \in \mathbb{Z}_{>0}$,

$$x^{2n}y = \frac{2n}{2n+1}x^{2n+1} + \frac{1}{2n+1}y^{2n+1} - S_{m,n}(x, y), \tag{V.3.1}$$

*where*

$$
\begin{aligned}
S_{m,n}(x,y) =& \frac{2^{m\lambda(n)}}{2^{m\lambda(n)}-1} \sum_{i=1}^{m\lambda(n)} 2^{-i}\chi_+(p_n(i)) x^{2n+1-p_n(i)} \left(x^{p_n(i-1)} - y^{p_n(i-1)}\right)^2 \\
&+ \frac{2^{m\lambda(n)}}{2^{m\lambda(n)}-1} \sum_{i=1}^{m\lambda(n)} 2^{-i}\chi_-(p_n(i)) y^{p_n(i)} \left(x^{2n+1-p_n(i-1)} - y^{2n+1-p_n(i-1)}\right)^2 . \qquad \text{(V.3.2)}
\end{aligned}
$$

**Remark 3.2.** *The identity (V.3.1) shows that $S_{m,n}(x,y)$ is the remainder of the Young inequality*

$$x^{2n}y \leq \frac{2n}{2n+1}x^{2n+1} + \frac{1}{2n+1}y^{2n+1}$$

*for all $x, y \geq 0$, where equality is attained if and only if $x = y$. See [3] for related subjects.*

**Remark 3.3.** *The identity (V.3.1) reduces to (V.2.8) [resp. (V.2.9)] when $x = 0$, $y = 1$ [resp. $x = 1$, $y = 0$].*

**Remark 3.4.** *All terms are of degree $2n+1$ as monomials of $x$ and $y$. Indeed, for $p_n(i)$ even, $p_n(i) = 2p_n(i-1)$ and therefore $2n+1-p_n(i)+2p_n(i-1) = 2n+1$, while for $p_n(i)$ odd, $p_n(i) = 2p_n(i-1) - 2n - 1$ and therefore $p_n(i) + 2(2n+1-p_n(i-1)) = 2n+1$.*

**Remark 3.5.** *In the cases where $n = 2^l - 1$ and $n = 2^{l-1}$ with $l \in \mathbb{Z}_{>0}$, by Remark 2.2, the identity (V.3.1) with $m = 1$ takes the following simple form:*

$$
\begin{aligned}
x^{2(2^l-1)}y =& \frac{2(2^l-1)}{2(2^l-1)+1}x^{2(2^l-1)+1} + \frac{1}{2(2^l-1)+1}y^{2(2l-1)+1} \\
&- \frac{2^{l+1}}{2(2^l-1)+1} \sum_{i=1}^{l} 2^{-i} x^{2^{l+1}-2^i-1} \left(x^{2^{i-1}} - y^{2^{i-1}}\right)^2 \\
&- \frac{1}{2(2^l-1)+1} y \left(x^{2^l-1} - y^{2^l-1}\right)^2 , \\
x^{2^l}y =& \frac{2^l}{2^l+1}x^{2^l+1} + \frac{1}{2^l+1}y^{2^l+1} \\
&- \frac{2^{2l}}{2^{2l}-1} \sum_{i=1}^{l} 2^{-i} x^{2^l-2^i+1} \left(x^{2^{i-1}} - y^{2^{i-1}}\right)^2 \\
&- \frac{2^l}{2^{2l}-1} \sum_{i=1}^{l} 2^{-i} y^{2^l-2^i+1} \left(x^{2^{i-1}} - y^{2^{i-1}}\right)^2 .
\end{aligned}
$$

*Proof of Proposition 3.1.* We expand terms in $S_{m,n}(x,y)$ and use identities in Proposition 2.3

and Remarks 2.4 and 3.3 to obtain

$$
\begin{aligned}
&S_{m,n}(x,y)\\
=&\frac{2^{m\lambda(n)}}{2^{m\lambda(n)}-1}\sum_{i=1}^{m\lambda(n)}2^{-i}\chi_+(p_n(i))\left(x^{2n+1}-2x^{2n+1-p_n(i-1)}y^{p_n(i-1)}+x^{2n+1-p_n(i)}y^{p_n(i)}\right)\\
&+\frac{2^{m\lambda(n)}}{2^{m\lambda(n)}-1}\sum_{i=1}^{m\lambda(n)}2^{-i}\chi_-(p_n(i))\left(x^{2n+1-p_n(i)}y^{p_n(i)}-2x^{2n+1-p_n(i-1)}y^{p_n(i-1)}+y^{2n+1}\right)\\
=&\frac{2^{m\lambda(n)}}{2^{m\lambda(n)}-1}\sum_{i=1}^{m\lambda(n)}2^{-i}\chi_+(p_n(i))x^{2n+1}+\frac{2^{m\lambda(n)}}{2^{m\lambda(n)}-1}\sum_{i=1}^{m\lambda(n)}2^{-i}\chi_-(p_n(i))y^{2n+1}\\
&-\frac{2^{m\lambda(n)}}{2^{m\lambda(n)}-1}\sum_{i=1}^{m\lambda(n)}2^{-i+1}\Big(\chi_+(p_n(i))+\chi_-(p_n(i))\Big)x^{2n+1-p_n(i-1)}y^{p_n(i-1)}\\
&+\frac{2^{m\lambda(n)}}{2^{m\lambda(n)}-1}\sum_{i=1}^{m\lambda(n)}2^{-i}\Big(\chi_+(p_n(i))+\chi_-(p_n(i))\Big)x^{2n+1-p_n(i)}y^{p_n(i)}\\
=&\frac{1}{2^{m\lambda(n)}-1}\sigma_n^+(m\lambda(n))x^{2n+1}+\frac{1}{2^{m\lambda(n)}-1}\sigma_n^-(m\lambda(n))y^{2n+1}\\
&+\frac{2^{m\lambda(n)}}{2^{m\lambda(n)}-1}\sum_{i=1}^{m\lambda(n)}\left(2^{-i}x^{2n+1-p_n(i)}y^{p_n(i)}-2^{-i+1}x^{2n+1-p_n(i-1)}y^{p_n(i-1)}\right)\\
=&\frac{2n}{2n+1}x^{2n+1}+\frac{1}{2n+1}y^{2n+1}\\
&+\frac{2^{m\lambda(n)}}{2^{m\lambda(n)}-1}\left(2^{-m\lambda(n)}x^{2n+1-p_n(m\lambda(n))}y^{p_n(m\lambda(n))}-x^{2n+1-p_n(0)}y^{p_n(0)}\right)\\
=&\frac{2n}{2n+1}x^{2n+1}+\frac{1}{2n}y^{2n+1}-x^{2n}y,
\end{aligned}
$$

as required. Q.E.D.

**Definition 3.6.** *For any $n\in\mathbb{Z}_{>0}$, $\nu_n:\mathbb{Z}_{>0}\to\mathbb{Z}_{>0}$ is defined as*

$$
\nu_n(j)=\begin{cases} j & \text{if} \quad j\le 2n+1,\\ j-2n-1 & \text{if} \quad j\ge 2n+2.\end{cases}
$$

**Remark 3.7.** *For any $n\in\mathbb{Z}_{>0}$,*

$$
\begin{aligned}
\sum_{i=1}^{2n+1}\sum_{j\neq i}x_i^{2n}x_j &= \sum_{j=1}^{2n+1}x_j\sum_{i=1}^{2n}x_{\nu_n(i+j)}^{2n},\\
\prod_{i=1}^{2n+1}x_i &= \frac{1}{2n+1}\sum_{j=1}^{2n+1}x_j\prod_{i=1}^{2n}x_{\nu_n(i+j)}.
\end{aligned}
$$

**Proposition 3.8.** *For any $m, n \in \mathbb{Z}_{>0}$,*

$$\sum_{i=1}^{2n+1} x_i^{2n+1} = \frac{1}{2n} \sum_{j=1}^{2n+1} x_j \sum_{i=1}^{2n} x_{\nu_n(i+j)}^{2n} + \frac{1}{2n} \sum_{i=1}^{2n+1} \sum_{j=1}^{2n+1} S_{m,n}(x_i, x_j), \tag{V.3.3}$$

*where $S_{m,n}(x, y)$ is as in (V.3.2).*

***proof***. By Proposition 3.1,

$$\begin{aligned}\sum_{i=1}^{2n+1} \sum_{j \neq i} x_i^{2n} x_j &= \sum_{i=1}^{2n+1} \sum_{j \neq i} \left( \frac{2n}{2n+1} x_i^{2n+1} + \frac{1}{2n+1} x_j^{2n+1} - S_{m,n}(x_i, x_j) \right) \\ &= \frac{(2n)^2}{2n+1} \sum_{i=1}^{2n+1} x_i^{2n+1} + \frac{2n}{2n+1} \sum_{j=1}^{2n+1} x_j^{2n+1} - \sum_{i=1}^{2n+1} \sum_{j \neq i} S_{m,n}(x_j, x_i) \\ &= 2n \sum_{i=1}^{2n+1} x_i^{2n+1} - \sum_{i=1}^{2n+1} \sum_{j=1}^{2n+1} S_{m,n}(x_i, x_j),\end{aligned}$$

since $S_{m,n}(x, x) = 0$. The proposition then follows from Remark 3.5. Q.E.D.

# 4 Main Theorems

In this section we state the main theorems. For that purpose we recall the argument of Hurwitz and Muirhead on symmetric means.

**Lemma 4.1** ( [14])**.** *For any $n \in \mathbb{Z}_{>0}$,*

$$\frac{1}{n} \sum_{i=1}^{n} x_i^n - \prod_{i=1}^{n} x_i = \frac{1}{2 \cdot n!} \sum_{j=1}^{n-1} \sum_{k=0}^{n-j-1} \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)}^{n-j-k-1} x_{\sigma(2)}^{k} \left( x_{\sigma(1)} - x_{\sigma(2)} \right)^2 \prod_{l=3}^{j+1} x_{\sigma(l)}.$$

***proof***. The lemma follows by writing

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} x_i^n - \prod_{i=1}^{n} x_i &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}^n - \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n} x_{\sigma(i)} \\
&= \frac{1}{n!}\sum_{j=1}^{n-1}\left(\sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}^{n-j+1}\prod_{l=2}^{j} x_{\sigma(l)} - \sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}^{n-j}\prod_{l=2}^{j+1} x_{\sigma(l)}\right) \\
&= \frac{1}{2\cdot n!}\sum_{j=1}^{n-1}\left(\sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}^{n-j+1}\prod_{l=3}^{j+1} x_{\sigma(l)} + \sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(2)}^{n-j+1}\prod_{l=3}^{j+1} x_{\sigma(l)}\right. \\
&\qquad\left. - \sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}^{n-j} x_{\sigma(2)}\prod_{l=3}^{j+1} x_{\sigma(i)} - \sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(2)}^{n-j} x_{\sigma(1)}\prod_{l=3}^{j+1} x_{\sigma(i)}\right) \\
&= \frac{1}{2\cdot n!}\sum_{j=1}^{n-1}\sum_{\sigma\in\mathfrak{S}_n}\left(x_{\sigma(1)}^{n-j} - x_{\sigma(2)}^{n-j}\right)\left(x_{\sigma(1)} - x_{\sigma(2)}\right)\prod_{l=3}^{j+1} x_{\sigma(l)} \\
&= \frac{1}{2\cdot n!}\sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1}\sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}^{n-j-k-1} x_{\sigma(2)}^{k}\left(x_{\sigma(1)} - x_{\sigma(2)}\right)^2\prod_{l=3}^{j+1} x_{\sigma(l)}.
\end{aligned}$$

Q.E.D.

The following identity provides a sum of squares formula in the even case. We write it down in the form that we use for the proof of the main theorem in the odd case.

**Theorem 1** ( [14])**.** *For any* $n \in \mathbb{Z}_{>0}$,

$$\begin{aligned}
&\frac{1}{2n}\sum_{i=1}^{2n} x_i^{2n} - \prod_{i=1}^{2n} x_i \\
=&\frac{1}{4\cdot n!}\sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1}\sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}^{2(n-j-k-1)} x_{\sigma(2)}^{2k}\left(x_{\sigma(1)}^2 - x_{\sigma(2)}^2\right)^2\prod_{l=3}^{j+1} x_{\sigma(l)}^2 \\
&+ \frac{1}{4\cdot n!}\sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1}\sum_{\sigma\in\mathfrak{S}_n} x_{n+\sigma(1)}^{2(n-j-k-1)} x_{n+\sigma(2)}^{2k}\left(x_{n+\sigma(1)}^2 - x_{n+\sigma(2)}^2\right)^2\prod_{l=3}^{j+1} x_{n+\sigma(l)}^2 \\
&+ \frac{1}{2}\left(\prod_{i=1}^{n} x_i - \prod_{i=1}^{n} x_{n+i}\right)^2.
\end{aligned}$$

***proof***. The theorem follows by applying Lemma 4.1 to the identity,

$$\begin{aligned}
\frac{1}{2n}\sum_{i=1}^{2n} x_i^{2n} - \prod_{i=1}^{2n} x_i =& \frac{1}{2}\left(\frac{1}{n}\sum_{i=1}^{n}(x_i^2)^n - \prod_{i=1}^{n} x_i^2\right) + \frac{1}{2}\left(\frac{1}{n}\sum_{i=1}^{n}(x_{n+i}^2)^n - \prod_{i=1}^{n} x_{n+i}^2\right) \\
&+ \frac{1}{2}\left(\prod_{i=1}^{n} x_i - \prod_{i=1}^{n} x_{n+i}\right)^2.
\end{aligned}$$

Q.E.D.

We are in a position to prove the main result.

**Theorem 2.** *For any $n \in \mathbb{Z}_{>0}$,*

$$
\begin{aligned}
&\frac{1}{2n+1}\sum_{i=1}^{2n+1} x_i^{2n+1} - \prod_{i=1}^{2n+1} x_i \\
=&\frac{1}{4(2n+1)\cdot n!}\sum_{i=1}^{2n+1} x_i \sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1}\sum_{\sigma\in\mathfrak{S}_n} x_{\nu_n(i+\sigma(1))}^{2(n-j-k-1)} x_{\nu_n(i+\sigma(2))}^{2k} \\
&\qquad\qquad\cdot\left(x_{\nu_n(i+\sigma(1))}^2 - x_{\nu_n(i+\sigma(2))}^2\right)^2 \prod_{l=3}^{j+1} x_{\nu_n(i+\sigma(l))}^2 \\
&+\frac{1}{4(2n+1)\cdot n!}\sum_{i=1}^{2n+1} x_i \sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1}\sum_{\sigma\in\mathfrak{S}_n} x_{\nu_n(i+n+\sigma(1))}^{2(n-j-k-1)} x_{\nu_n(i+n+\sigma(2))}^{2k} \\
&\qquad\qquad\cdot\left(x_{\nu_n(i+n+\sigma(1))}^2 - x_{\nu_n(i+n+\sigma(2))}^2\right)^2 \prod_{l=3}^{j+1} x_{\nu_n(i+n+\sigma(l))}^2 \\
&+\frac{1}{2(2n+1)}\sum_{i=1}^{2n+1} x_i\left(\prod_{j=1}^{n} x_{\nu_n(i+j)} - \prod_{j=1}^{n} x_{\nu_n(i+n+j)}\right)^2 \\
&+\frac{1}{2n(2n+1)}\sum_{i=1}^{2n+1}\sum_{j=1}^{2n+1} S_{1,n}(x_i,x_j),
\end{aligned}
\tag{V.4.1}
$$

*where $S_{1,n}(x,y)$ is as in (V.3.2) with $m=1$.*

***proof***. By Remark 3.5 and Proposition 3.2, we have

$$
\begin{aligned}
&\frac{1}{2n+1}\sum_{i=1}^{2n+1} x_i^{2n+1} - \prod_{i=1}^{2n+1} x_i \\
&=\frac{1}{2n+1}\sum_{i=1}^{2n+1} x_i\left(\frac{1}{2n}\sum_{j=1}^{2n} x_{\nu_n(i+j)}^{2n} - \prod_{j=1}^{2n} x_{\nu_n(i+j)}\right) + \frac{1}{2n(2n+1)}\sum_{i=1}^{2n+1}\sum_{j=1}^{2n+1} S_{1,n}(x_i,x_j).
\end{aligned}
$$

Then, the theorem follows from Theorem 4.1. Q.E.D.

## 5 Examples

In this section, we show some explicit formulae of Theorem 4.2. For $n \in \mathbb{Z}_{>0}$, we define $R_1(n)$ and $R_2(n)$ as follows;

$$R_1(n) = \frac{1}{4(2n+1)\cdot n!} \sum_{i=1}^{2n+1} x_i \sum_{j=1}^{n-1} \sum_{k=0}^{n-j-1} \sum_{\sigma \in \mathfrak{S}_n} x_{\nu_n(i+\sigma(1))}^{2(n-j-k-1)} x_{\nu_n(i+\sigma(2))}^{2k},$$

$$R_2(n) = \frac{1}{4(2n+1)\cdot n!} \sum_{i=1}^{2n+1} x_i \sum_{j=1}^{n-1} \sum_{k=0}^{n-j-1} \sum_{\sigma \in \mathfrak{S}_n} x_{\nu_n(i+n+\sigma(1))}^{2(n-j-k-1)} x_{\nu_n(i+n+\sigma(2))}^{2k}.$$

$R_1(n)$ is the first term on RHS of (V.4.1) and $R_2(n)$ is the second one.

1. The case of 3 variables.

In this case, $R_1(1)$ and $R_2(1)$ are equal to 0 and

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{3} S_{1,1}(x_i, x_j) &= \frac{4}{3}\sum_{i=1}^{3}\sum_{j=1}^{3} \frac{1}{2}x_i(x_i - x_j)^2 + \frac{1}{4}x_j(x_i - x_j)^2 \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} x_i(x_i - x_j)^2.\end{aligned}$$

Then,

$$\frac{1}{3}\sum_{i=1}^{3} x_i^3 - \prod_{i=1}^{3} x_i = \frac{1}{6}(x_1 + x_2 + x_3)\Big\{(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2\Big\}.$$

2. The case of 5 variables.

$R_1(2)$ and $R_2(2)$ are computed as

$$R_1(2) = \frac{1}{4\cdot 5\cdot 2} \sum_{i=1}^{5} x_i \left[ \left(x_{\nu_n(i+1)}^2 - x_{\nu_n(i+2)}^2\right)^2 + \left(x_{\nu_n(i+2)}^2 - x_{\nu_n(i+1)}^2\right)^2 \right],$$

$$R_2(2) = \frac{1}{20} \sum_{i=1}^{5} x_i \left[ \left(x_{\nu_n(i+3)}^2 - x_{\nu_n(i+4)}^2\right)^2 \right].$$

In addition,

$$\begin{aligned}
&\frac{1}{20}\sum_{i=1}^{5}\sum_{j=1}^{5}S_{1,2}(x_i,x_j)\\
&=\frac{1}{20}\cdot\frac{16}{15}\sum_{i=1}^{5}\sum_{j=1}^{5}\left\{\frac{1}{2}x_i^3(x_i-x_j)^2+\frac{1}{4}x_i(x_i^2-x_j^2)^2+\frac{1}{8}x_j^3(x_i-x_j)^2+\frac{1}{16}x_j(x_i^2-x_j^2)^2\right\}\\
&=\frac{1}{30}\sum_{i=1}^{5}\sum_{j=1}^{5}x_i^3(x_i-x_j)^2+\frac{1}{60}\sum_{i=1}^{5}\sum_{j=1}^{5}x_i^3(x_i-x_j)^2.
\end{aligned}$$

Then,

$$\begin{aligned}
&\frac{1}{5}\sum_{i=1}^{5}x_i^5-\prod_{i=1}^{5}x_i\\
=&\frac{1}{20}x_1\Big\{(x_2^2-x_3^2)^2+(x_4^2-x_5^2)^2+2(x_2x_3-x_4x_5)^2\Big\}\\
&+\frac{1}{20}x_2\Big\{(x_3^2-x_4^2)^2+(x_5^2-x_1^2)^2+2(x_3x_4-x_5x_1)^2\Big\}\\
&+\frac{1}{20}x_3\Big\{(x_4^2-x_5^2)^2+(x_1^2-x_2^2)^2+2(x_4x_5-x_1x_2)^2\Big\}\\
&+\frac{1}{20}x_4\Big\{(x_5^2-x_1^2)^2+(x_2^2-x_3^2)^2+2(x_5x_1-x_2x_3)^2\Big\}\\
&+\frac{1}{20}x_5\Big\{(x_1^2-x_2^2)^2+(x_3^2-x_4^2)^2+2(x_1x_2-x_3x_4)^2\Big\}\\
&+\frac{1}{30}\sum_{i=1}^{5}\sum_{j=1}^{5}x_i^3(x_i-x_j)^2+\frac{1}{60}\sum_{i=1}^{5}\sum_{j=1}^{5}x_i(x_i^2-x_j^2)^2.
\end{aligned}$$

3. The case of 7 variables.

We compute

$$
\begin{aligned}
&R_1(3) \\
&= \frac{1}{168} \sum_{i=1}^{7} x_i \sum_{\sigma \in \mathfrak{S}_3} \left( x^2_{\nu_3(i+\sigma(1))} + x^2_{\nu_3(i+\sigma(2))} + x^2_{\nu_3(i+\sigma(3))} \right) \left( x^2_{\nu_3(i+\sigma(1))} - x^2_{\nu_3(i+\sigma(2))} \right)^2 \\
&= \frac{1}{168} \sum_{i=1}^{7} x_i \left( x^2_{\nu_3(i+1)} + x^2_{\nu_3(i+2)} + x^2_{\nu_3(i+3)} \right) \sum_{\sigma \in \mathfrak{S}_3} \left( x^2_{\nu_3(i+\sigma(1))} - x^2_{\nu_3(i+\sigma(2))} \right)^2 \\
&= \frac{1}{84} \sum_{i=1}^{7} x_i \left( x^2_{\nu_3(i+1)} + x^2_{\nu_3(i+2)} + x^2_{\nu_3(i+3)} \right) \\
&\qquad \cdot \left\{ \left( x^2_{\nu_3(i+1)} - x^2_{\nu_3(i+2)} \right)^2 + \left( x^2_{\nu_3(i+2)} - x^2_{\nu_3(i+3)} \right)^2 + \left( x^2_{\nu_3(i+3)} - x^2_{\nu_3(i+1)} \right)^2 \right\}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\frac{1}{7}\sum_{i=1}^{7} x_i^7 - \prod_{i=1}^{7} x_i \\
=&\frac{x_1}{84}\Bigg[(x_2^2+x_3^2+x_4^2)\Big\{(x_2^2-x_3^2)^2+(x_3^2-x_4^2)^2+(x_4^2-x_2^2)^2\Big\} \\
&\qquad + (x_5^2+x_6^2+x_7^2)\Big\{(x_5^2-x_6^2)^2+(x_6^2-x_7^2)^2+(x_7^2-x_5^2)^2\Big\} + 6(x_2x_3x_4-x_5x_6x_7)^2\Bigg] \\
&+\frac{x_2}{84}\Bigg[(x_3^2+x_4^2+x_5^2)\Big\{(x_3^2-x_4^2)^2+(x_4^2-x_5^2)^2+(x_5^2-x_3^2)^2\Big\} \\
&\qquad + (x_6^2+x_7^2+x_1^2)\Big\{(x_6^2-x_7^2)^2+(x_7^2-x_1^2)^2+(x_1^2-x_6^2)^2\Big\} + 6(x_3x_4x_5-x_6x_7x_1)^2\Bigg] \\
&+\frac{x_3}{84}\Bigg[(x_4^2+x_5^2+x_6^2)\Big\{(x_4^2-x_5^2)^2+(x_5^2-x_6^2)^2+(x_6^2-x_4^2)^2\Big\} \\
&\qquad + (x_7^2+x_1^2+x_2^2)\Big\{(x_7^2-x_1^2)^2+(x_1^2-x_2^2)^2+(x_2^2-x_7^2)^2\Big\} + 6(x_4x_5x_6-x_7x_1x_2)^2\Bigg] \\
&+\frac{x_4}{84}\Bigg[(x_5^2+x_6^2+x_7^2)\Big\{(x_5^2-x_6^2)^2+(x_6^2-x_7^2)^2+(x_7^2-x_5^2)^2\Big\} \\
&\qquad + (x_1^2+x_2^2+x_3^2)\Big\{(x_1^2-x_2^2)^2+(x_2^2-x_3^2)^2+(x_3^2-x_1^2)^2\Big\} + 6(x_5x_6x_7-x_1x_2x_3)^2\Bigg] \\
&+\frac{x_5}{84}\Bigg[(x_6^2+x_7^2+x_1^2)\Big\{(x_6^2-x_7^2)^2+(x_7^2-x_1^2)^2+(x_1^2-x_6^2)^2\Big\} \\
&\qquad + (x_2^2+x_3^2+x_4^2)\Big\{(x_2^2-x_3^2)^2+(x_3^2-x_4^2)^2+(x_4^2-x_2^2)^2\Big\} + 6(x_6x_7x_1-x_2x_3x_4)^2\Bigg] \\
&+\frac{x_6}{84}\Bigg[(x_7^2+x_1^2+x_2^2)\Big\{(x_7^2-x_1^2)^2+(x_1^2-x_2^2)^2+(x_2^2-x_7^2)^2\Big\} \\
&\qquad + (x_3^2+x_4^2+x_5^2)\Big\{(x_3^2-x_4^2)^2+(x_4^2-x_5^2)^2+(x_5^2-x_3^2)^2\Big\} + 6(x_7x_1x_2-x_3x_4x_5)^2\Bigg] \\
&+\frac{x_7}{84}\Bigg[(x_1^2+x_2^2+x_3^2)\Big\{(x_1^2-x_2^2)^2+(x_2^2-x_3^2)^2+(x_3^2-x_1^2)^2\Big\} \\
&\qquad + (x_4^2+x_5^2+x_6^2)\Big\{(x_4^2-x_5^2)^2+(x_5^2-x_6^2)^2+(x_6^2-x_4^2)^2\Big\} + 6(x_1x_2x_3-x_4x_5x_6)^2\Bigg] \\
&+\frac{2}{147}\sum_{i=1}^{7}\sum_{j=1}^{7}\left\{x_i^5(x_i-x_j)^2+\frac{1}{2}x_i^3(x_i^2-x_j^2)^2+\frac{1}{4}x_i(x_i^3-x_j^3)^2\right\}.
\end{aligned}
$$

4. The case of 9 variables.

$R_1(4)$ is computed as

$$R_1(4) = \frac{1}{864}\sum_{i=1}^{9} x_i \sum_{\sigma\in\mathfrak{S}_4} \left(x_{\nu_4(i+\sigma(1))}^2 - x_{\nu_4(i+\sigma(2))}^2\right)^2$$
$$\cdot \Big\{ x_{\nu_4(i+\sigma(1))}^4 + x_{\nu_4(i+\sigma(1))}^2 x_{\nu_4(i+\sigma(2))}^2 + x_{\nu_4(i+\sigma(1))}^4 + x_{\nu_4(i+\sigma(3))}^2 x_{\nu_4(i+\sigma(4))}^2$$
$$+ (x_{\nu_4(i+\sigma(1))}^2 + x_{\nu_4(i+\sigma(2))}^2) x_{\nu_4(i+\sigma(3))}^2 \Big\}.$$

In addition

$$\frac{1}{8\cdot 9}\sum_{i=1}^{9}\sum_{j=1}^{9} S_{1,4}(x_i, x_j)$$
$$= \frac{8}{567}\sum_{i=1}^{9}\sum_{j=1}^{9}\Big\{ 2^{-1}x_i^7(x_i - x_j)^2 + 2^{-2}x_i^5(x_i^2 - x_j^2)^2 + 2^{-3}x_i(x_i^4 - x_j^4)^2$$
$$+ 2^{-4}x_j^7(x_i - x_j)^2 + 2^{-5}x_j^5(x_i^2 - x_j^2)^2 + 2^{-6}x_j(x_i^4 - x_j^4)^2 \Big\}$$
$$= \frac{1}{63}\sum_{i=1}^{9}\sum_{j=1}^{9}\left\{ \frac{1}{2}x_i^7(x_i - x_j)^2 + \frac{1}{4}x_i^5(x_i^2 - x_j^2)^2 + \frac{1}{8}x_i(x_i^4 - x_j^4)^2 \right\}.$$

Then,

$$\frac{1}{9}\sum_{i=1}^{9} x_i^9 - \prod_{i=1}^{9} x_i$$
$$= \frac{1}{864}\sum_{i=1}^{9} x_i \Big\{ P(x_{\nu_4(i+1)},\ x_{\nu_4(i+2)},\ x_{\nu_4(i+3)},\ x_{\nu_4(i+4)}\ )$$
$$+ P(x_{\nu_4(i+1)},\ x_{\nu_4(i+3)},\ x_{\nu_4(i+4)},\ x_{\nu_4(i+2)}\ ) + P(x_{\nu_4(i+1)},\ x_{\nu_4(i+4)},\ x_{\nu_4(i+2)},\ x_{\nu_4(i+3)}\ )$$
$$+ P(x_{\nu_4(i+2)},\ x_{\nu_4(i+3)},\ x_{\nu_4(i+4)},\ x_{\nu_4(i+1)}\ ) + P(x_{\nu_4(i+2)},\ x_{\nu_4(i+4)},\ x_{\nu_4(i+1)},\ x_{\nu_4(i+3)}\ )$$
$$+ P(x_{\nu_4(i+3)},\ x_{\nu_4(i+4)},\ x_{\nu_4(i+1)},\ x_{\nu_4(i+2)}\ ) + P(x_{\nu_4(i+5)},\ x_{\nu_4(i+6)},\ x_{\nu_4(i+7)},\ x_{\nu_4(i+8)}\ )$$
$$+ P(x_{\nu_4(i+5)},\ x_{\nu_4(i+7)},\ x_{\nu_4(i+8)},\ x_{\nu_4(i+6)}\ ) + P(x_{\nu_4(i+5)},\ x_{\nu_4(i+8)},\ x_{\nu_4(i+6)},\ x_{\nu_4(i+7)}\ )$$
$$+ P(x_{\nu_4(i+6)},\ x_{\nu_4(i+7)},\ x_{\nu_4(i+8)},\ x_{\nu_4(i+5)}\ ) + P(x_{\nu_4(i+6)},\ x_{\nu_4(i+8)},\ x_{\nu_4(i+5)},\ x_{\nu_4(i+7)}\ )$$
$$+ P(x_{\nu_4(i+7)},\ x_{\nu_4(i+8)},\ x_{\nu_4(i+5)},\ x_{\nu_4(i+6)}\ ) \Big\}$$
$$+ \frac{1}{18}\sum_{i=1}^{9} x_i \left( \prod_{j=1}^{4} x_{\nu_4(i+j)} - \prod_{j=5}^{8} x_{\nu_4(i+j)} \right)^2$$
$$+ \frac{1}{63}\sum_{i=1}^{9}\sum_{j=1}^{9}\left\{ \frac{1}{2}x_i^7(x_i - x_j)^2 + \frac{1}{4}x_i^5(x_i^2 - x_j^2)^2 + \frac{1}{8}x_i(x_i^4 - x_j^4)^2 \right\},$$

where

$$P(x,y,z,w) = (x^2-y^2)^2\Big\{4(x^4+x^2y^2+y^4+z^2w^2)+2(x^2+y^2)(z^2+w^2)\Big\}.$$

## References


[1] R. Aboudi and D. Thon, “Refinements of Muirhead’s lemma and income inequality,” Math. Social Sci., **51**(2006), 201-216.

[2] J.M. Aldaz, “A refinement of the inequality between arithmetic and geometric means,” J. Math. Inequal., **2**(2008), 473–477.

[3] J.M. Aldaz, “Self-improvement of the inequality between arithmetic and geometric means,” J. Math. Inequal., **3**(2009), 213–216.

[4] J.M. Aldaz, “Sharp bounds for the difference between the arithmetic and geometric means,” Arch. Math. (Basel), **99**(2012), 393–399.

[5] H. Alzer, “A new refinement of the arithmetic mean–geometric meaninequality,” Rocky Mountain J. Math., **27**(1997), 663–667.

[6] P. Bracken, “An arithmetic-geometric mean inequality,” Expo. Math., **19**(2001), 273–279.

[7] A. Cuttler, C. Greene and M. Skandera, “Inequalities for symmetric means,” European J. Combin., **32**(2011), 745–761.

[8] Z. Cvetkovski, “Inequalities,” Springer, Heidelberg, 2012.

[9] C. Fidalgo and A. Kovacec, “Positive semidefinite diagonal minus tail forms are sums of squares,” Math. Z., **269**(2011), 629–645.

[10] P.E. Frenkel and P. Horváth, “Minkowskis inequality and sums of squares,” Cent. Eur. J. Math., **12** (2014).

[11] R. Fujisawa, “Algebraic means,” Proc. Imp. Acad., **1**(1918), 159–170.

[12] K. Fujiwara and T. Ozawa, “Exact Remainder Formula for the Young Inequality and Applications,” Int. J. Math. Anal., **7**(2013), 2723–2735.

[13] M. Ghasemi and M. Marshall, “Lower bounds for polynomials using geometric programming,” SIAM J. Optim., **22**(2012), 460–473.

[14] G.H. Hardy, J.E. Littlewood and G. Polya, “Inequalities,” 2nd ed., Cambridge University Press, 1988.

[15] A. Hurwitz, “Über den Vergleich des arithmetischen und des geometrischen Mittels,” J. reine angew. Math., **108**(1891), 266–268.

[16] R.F. Muirhead, “Some methods applicable to identities and inequalities of symmetric algebraic functions of $n$ letters,” Proc. Edinburgh Math. Soc., **21**(1902), 144–162.

[17] M.R. Peter , “Refined arithmetic, geometric and harmonic mean inequalities,” Rocky Mountain J. Math., **33**(2003), 1459–1464.

[18] J.V. Ryff, “On Muirhead’s theorem,” Pacific J. Math., **21**(1967), 567–576.

[19] T.V. Tararykova, “An explicit representation as quasi-sum of squares of a polynomial generated by the AG inequality,” Math. Inequal. Appl., **9**(2006), 649–659.

[20] M.K. Vamanamurthy and M.Vuorinen, “Inequalities for means,” J. Math. Anal. Appl., **183**(1994), 155–166.

## Part VI

# Appendix C: Stability of the Young and Hölder inequalities

**Abstract.** We give a simple proof of Aldaz' stability version of the Young and Hölder inequalities and further refinements of available stability versions of those inequalities.

2000 Mathematics Subject Classification: 26D15

Keywords: Young's inequality, Hölder's inequality

## 1 Introduction

In this paper, we study the Young and Hölder inequalities from the point of view of the deviation from equalities with better upper and lower bound estimates. Particularly, we give a further refinement of Aldaz' stability type inequalities [1] as well as a simple proof based exclusively on an algebraic argument with the standard Young inequality.

Throughout this paper, the following remainder function [7] plays an important role:

$$R(\theta; a, b) = \theta a + (1-\theta)b - a^{\theta}b^{1-\theta}, \tag{VI.1.1}$$

where $a, b > 0$ and $0 \leq \theta \leq 1$.

The standard Young inequality is described as

$$R(\theta; a, b) \geq 0, \tag{VI.1.2}$$

which may be used without particular comments. The standard Hölder inequality follows from (VI.1.2) and the equality

$$\int_{\Omega} |fg| d\mu = \|f\|_p \|g\|_{p'} \left(1 - \int_{\Omega} R\left(\frac{1}{p}; \frac{|f|^p}{\|f\|_p^p}, \frac{|g|^{p'}}{\|g\|_{p'}^{p'}}\right) d\mu\right) \tag{VI.1.3}$$

for all $f \in L^p(\Omega, \mu)\backslash\{0\}$ and $g \in L^{p'}(\Omega, \mu)\backslash\{0\}$, where $L^q(\Omega, \mu)$ is the Banach space of $q$-th integrable functions on a measure space $(\Omega, \mu)$ with norm $\|\cdot\|_q$, $1 < q < \infty$, and $p'$ is the dual exponent of $p$ defined by $1/p + 1/p' = 1$.

The purpose in this paper is to give a clear understanding of the standard Young and Hölder inequalities on the basis of upper and lower bound estimates on the remainder function $R(\theta; a, b)$. In Section 2, we reexamine the multiplication formula on $R(\theta; a, b)$ [7] and present its dual formula. As a corollary, we give an algebraic proof of Aldaz' stability type inequalities for the Young and Hölder inequalities [1]. In Section 3, we compare upper and lower bound estimates on $R(\theta; a, b)$ in [1,3,6,7]. In Section 4, we give dyadic refinements of the multiplication formulae on $R(\theta; a, b)$ with their straightforward corollaries on (VI.1.3) and discuss the associated dyadic refinements of the Hölder inequality.

There are many papers on the related subjects. We refer the reader to [1,3–8] and references therein.

We close the introduction by giving some notation to be used in this paper. For $a, b \in \mathbb{R}$ we denote by $a \wedge b$ and $a \vee b$ their minimum and maximum, respectively.

## 2 Multiplication Formulae

In this section, we revisit the original multiplication formula on $R(\theta; a, b)$ [7] in connection with Aldaz' stability type inequalities [1]. First of all, we recall Kichenassamy's multiplication formula.

**Proposition 2.1** (Kichenassamy [7]). *Let $\theta$ and $\sigma$ satisfy $0 \leq \theta, \sigma \leq 1$. Then, the equality*

$$R(\theta\sigma; a, b) = \theta R(\sigma; a, b) + b^{1-\sigma} R(\theta; a^{\sigma}, b^{\sigma}) \tag{VI.2.1}$$

*holds for all $a, b > 0$.*

***proof***. The proposition follows from the equality

$$\begin{aligned} R(\sigma\theta, a, b) &= \sigma\theta a + (1-\theta\sigma)b - a^{\sigma\theta} b^{1-\sigma\theta} \\ &= \theta\big(\sigma a + (1-\sigma)b - a^{\sigma} b^{1-\sigma}\big) + \theta a^{\sigma} b^{1-\sigma} + (1-\theta)b - a^{\sigma\theta} b^{1-\sigma\theta} \\ &= \theta R(\sigma, a, b) + b^{1-\sigma}\big(\theta a^{\sigma} + (1-\theta)b^{\sigma} - a^{\sigma\theta} b^{\sigma(1-\theta)}\big). \end{aligned}$$

Q.E.D.

**Corollary 2.2.** *Let $\theta$ and $\sigma$ satisfy $0 < \theta \leq \sigma < 1$. Then, the equality*

$$R(\theta; a, b) = \frac{\theta}{\sigma} R(\sigma; a, b) + b^{1-\sigma} R\Big(\frac{\theta}{\sigma}; a^{\sigma}, b^{\sigma}\Big) \tag{VI.2.2}$$

*holds for all $a, b > 0$.*

**Proposition 2.3.** *Let $\theta$ and $\sigma$ satisfy $0 \leq \theta \leq \sigma < 1$. Then, the equality*

$$R(\sigma, a, b) = \frac{1-\sigma}{1-\theta} R(\theta, a, b) + a^{\theta} R\left(\frac{\sigma-\theta}{1-\theta}, a^{1-\theta}, b^{1-\theta}\right). \tag{VI.2.3}$$

*holds for all $a, b > 0$.*

**Remark 2.4.** *The equality* (VI.2.3) *is regarded as a dual formula for $R(\theta; a, b)$ in the sense that* $\dfrac{1-\sigma}{1-\theta} + \dfrac{\sigma-\theta}{1-\theta} = 1$.

*Proof of Proposition 2.3.*

$$\begin{aligned} &R(\sigma, a, b) \\ &= \sigma a + (1-\sigma)b - a^{\sigma} b^{1-\sigma} \\ &= \frac{1-\sigma}{1-\theta}\Big(\theta a + (1-\theta)b - a^{\theta} b^{1-\theta}\Big) + \frac{\sigma-\theta}{1-\theta} a + \frac{1-\sigma}{1-\theta} a^{\theta} b^{1-\theta} - a^{\sigma} b^{1-\sigma} \\ &= \frac{1-\sigma}{1-\theta} R(\theta, a, b) + a^{\theta}\Big(\frac{\sigma-\theta}{1-\theta} a^{1-\theta} + \frac{1-\sigma}{1-\theta} b^{1-\theta} - a^{\sigma-\theta} b^{(1-\theta)\left(1-\frac{\sigma-\theta}{1-\theta}\right)}\Big). \end{aligned}$$

Q.E.D.

**Corollary 2.5.** *Let $\theta$ and $\sigma$ satisfy $0 \le \theta \le \sigma < 1$. Then, the equality*

$$R(\theta; a, b) = \frac{1-\theta}{1-\sigma} R(\sigma; a, b) - \frac{1-\theta}{1-\sigma} a^{\theta} R\Big(\frac{\sigma-\theta}{1-\theta}; a^{1-\theta}, b^{1-\theta}\Big) \tag{VI.2.4}$$

*holds for all $a, b > 0$.*

**Remark 2.6.** *Propositions 2.1 and 2.3 are equivalent. In fact, it follows form the reciprocal formula $R(\theta; a, b) = R(1-\theta, b, a)$ and Proposition 2.1 that*

$$\begin{aligned} R(\sigma; a, b) &= R(1-\sigma; b, a) \\ &= \frac{1-\sigma}{1-\theta} R(1-\theta; b, a) + a^{1-(1-\theta)} R\Big(\frac{1-\sigma}{1-\theta}; b^{1-\theta}, a^{1-\theta}\Big) \\ &= \frac{1-\sigma}{1-\theta} R(\theta; a, b) + a^{\theta} R\Big(\frac{\sigma-\theta}{1-\theta}; a^{1-\theta}, b^{1-\theta}\Big), \end{aligned}$$

*which is precisely* (VI.2.3)*. Conversely, given $\theta$ and $\sigma$ with $0 < \theta \le 1$, $0 < \sigma \le 1$, we put $\theta' = 1 - \theta\sigma$ and $\sigma' = 1 - \sigma$. Then, we have $0 \le \sigma' \le \theta' < 1$, $\sigma = 1 - \sigma'$, $\theta = (1-\theta')/(1-\sigma')$, and $\theta\sigma = 1 - \theta'$. By the reciprocal formula and Proposition 2.3, we have*

$$\begin{aligned} R(\theta\sigma; a, b) &= R(1-\theta'; a, b) = R(\theta'; b, a) \\ &= \frac{1-\theta'}{1-\sigma'} R(\sigma'; b, a) + b^{\sigma'} R\Big(\frac{\theta'-\sigma'}{1-\sigma'}; b^{1-\sigma'}, a^{1-\sigma'}\Big) \\ &= \frac{1-\theta'}{1-\sigma'} R(1-\sigma'; a, b) + b^{\sigma'} R\Big(\frac{1-\theta'}{1-\sigma'}; a^{1-\sigma'}, b^{1-\sigma'}\Big) \\ &= \theta R(\sigma; a, b) + b^{1-\sigma} R(\theta; a^{\sigma}, b^{\sigma}) \end{aligned}$$

*which is precisely* (VI.2.1)*.*

**Proposition 2.7** (Aldaz [1], Kichenassamy [7])**.** *Let $0 \le \theta \le 1$. Then, the inequalities*

$$\big(\theta \wedge (1-\theta)\big)(a^{1/2} - b^{1/2})^2 \le R(\theta; a, b) \le \big(\theta \vee (1-\theta)\big)(a^{1/2} - b^{1/2})^2 \tag{VI.2.5}$$

*hold for all $a, b > 0$.*

***proof***. Though the first inequality of (VI.2.5) is shown in [7], we show the inequalities in (VI.2.5) for completeness. In the case $0 \le \theta \le 1/2$, we use Corollaries 2.2 and 2.5 with $\sigma = 1/2$ to obtain

$$\begin{aligned} \theta(a^{1/2} - b^{1/2})^2 &= 2\theta R(1/2; a, b) = R(\theta; a, b) - b^{1/2} R(2\theta; a, b) \\ &\le R(\theta; a, b) \\ &= 2(1-\theta) R(1/2; a, b) - 2(1-\theta) a^{\theta} R\Big(\frac{1/2-\theta}{1-\theta}; a^{1-\theta}, b^{1-\theta}\Big) \\ &\le 2(1-\theta) R(1/2; a, b) = (1-\theta)(a^{1/2} - b^{1/2})^2. \end{aligned} \tag{VI.2.6}$$

In the case $1/2 \le \theta \le 1$, we apply (VI.2.6) with $\theta$ replaced by $1-\theta$ to obtain

$$2(1-\theta) R(1/2; b, a) \le R(1-\theta; b, a) \le 2\theta R(1/2; b, a),$$

which is precisely (VI.2.5). Q.E.D.

**Remark 2.8.** *An equivalent couple of inequalities in Proposition 2.7 was proved by Aldaz [1] by differential calculus. The proof above depends on algebraic identities with the standard Young inequality.*

## 3 Upper and Lower Bounds of the Remainder Function

In this section, we collect and compare several bounds of the remainder function $R(\theta; a, b)$. For that purpose, we study the upper and lower bound estimates in terms of majorant $M(\theta; a, b)$ and minorant $m(\theta; a, b)$ in the form

$$m(\theta; a, b) \leq R(\theta; a, b) \leq M(\theta; a, b)$$

for all $a, b > 0$. We introduce four couples of the bounds:

$$\begin{aligned}
\text{[A]} \quad & m_A(\theta; a, b) = \big(\theta \wedge (1-\theta)\big)(a^{1/2} - b^{1/2})^2, \\
& M_A(\theta; a, b) = \big(\theta \vee (1-\theta)\big)(a^{1/2} - b^{1/2})^2. \\
\text{[K]} \quad & m_K(\theta; a, b) = \frac{\theta(1-\theta)}{2}(a \wedge b)(\log a - \log b)^2, \\
& M_K(\theta; a, b) = \frac{\theta(1-\theta)}{2}(a \vee b)(\log a - \log b)^2. \\
\text{[H]} \quad & m_H(\theta; a, b) = \Big(\theta \wedge (1-\theta)\Big)\Big|a^{\theta\wedge(1-\theta)} - b^{\theta\wedge(1-\theta)}\Big|^{1/(\theta\wedge(1-\theta)}, \\
& M_H(\theta; a, b) = \Big(\theta \vee (1-\theta)\Big)\Big|a^{\theta\vee(1-\theta)} - b^{\theta\vee(1-\theta)}\Big|^{1/(\theta\vee(1-\theta)}. \\
\text{[FO]} \quad & m_{FO}(\theta; a, b) = \frac{\theta(1-\theta)}{2(a \vee b)}(a-b)^2, \\
& M_{FO}(\theta; a, b) = \frac{\theta(1-\theta)}{2(a \wedge b)}(a-b)^2.
\end{aligned}$$

Those couples are given respectively in [1], [7], [6], and [3].

**Remark 3.1.** *By the monotonicity property suggested in [7], the remainder function with respect to $\theta \in [0, 1]$ is approximated arbitrary precisely by the remainder functions with respect to rationals which approximate $\theta$. However, the approximation obtained by the monotonicity property is rather involved. Here, we focus only lower and upper bounds with regard to a difference.*

Simple relationships in those couples are summarized in:

**Proposition 3.2.** *Let $0 \leq \theta \leq 1$. Then, the inequalities*

$$m_H(\theta; a, b) \leq m_A(\theta; a, b) \leq R(\theta, a, b) \leq M_A(\theta; a, b) \leq M_H(\theta, a, b), \quad \text{(VI.3.1)}$$

$$m_K(\theta; a, b) \leq m_{FO}(\theta; a, b) \leq R(\theta, a, b) \leq M_K(\theta; a, b) \leq M_{FO}(\theta, a, b) \quad \text{(VI.3.2)}$$

*hold for all $a, b > 0$.*

***proof***. By homogeneity, (VI.3.1) follow from the inequality

$$(x^\theta - 1)^{1/\theta} \le (x^\sigma - 1)^{1/\sigma} \tag{VI.3.3}$$

for all $x \ge 1$ and any $\theta$ and $\sigma$ with $0 \le \theta \le \sigma$. The inequality (VI.3.3) follows from

$$x^\theta = (x^\sigma - 1 + 1)^{\theta/\sigma} \le (x^\sigma - 1)^{\theta/\sigma} + 1.$$

Although some inequalities in (VI.3.2) are proved in [2,6], we prove (VI.3.2) for completeness. By the integral representations [2,6]

$$\begin{aligned} R(\theta; a, b) &= \theta(1-\theta)\left[\int_0^1 \int_0^t (ta + (1-t)b)^{\theta-1}(sa + (1-s)b)^{-\theta} ds dt\right](a-b)^2 \\ &= \left[\int_0^1 \Big(\big(\theta(1-t)\big) \wedge \big((1-\theta)t\big)\Big) a^t b^{1-t} dt\right](\log a - \log b)^2, \end{aligned}$$

we have

$$\begin{aligned} m_{FO}(\theta; a, b) \le R(\theta; a, b) \le M_{FO}(\theta; a, b), \\ m_K(\theta; a, b) \le R(\theta; a, b) \le M_K(\theta; a, b). \end{aligned}$$

Then, it suffices to prove that

$$\begin{aligned} m_K(\theta; a, b) \le m_{FO}(\theta; a, b), \\ M_K(\theta; a, b) \le M_{FO}(\theta; a, b). \end{aligned}$$

The last two inequalities are equivalent and follow from

$$x(\log x)^2 \le (x-1)^2$$

for all $x > 0$. Q.E.D.

**Proposition 3.3.** *Let* $0 < \theta < 1$ *and let*

$$t_0(\theta) = \left(\sqrt{\frac{2}{(\theta \vee (1-\theta))}} - 1\right)^2.$$

*Then, the following inequalities hold for all* $a, b > 0$*:*

$$m_A(\theta, a, b) \le m_{FO}(\theta, a, b) \quad \text{if} \quad (a \vee b)\ t_0(\theta) \le a \wedge b, \tag{VI.3.4}$$

$$m_A(\theta, a, b) \ge m_{FO}(\theta, a, b) \quad \text{if} \quad 0 < a \wedge b \le (a \vee b)\ t_0(\theta). \tag{VI.3.5}$$

**Remark 3.4.** *Since* $0 < \theta \wedge (1-\theta) \le 1/2 \le \theta \vee (1-\theta) < 1$, $t_0(\theta)$ *satisfies*

$$(\sqrt{2} - 1)^2 < t_0(\theta) \le 1$$

*for all* $\theta$ *with* $0 \le \theta \le 1$*. Proposition 3.3 shows that* $m_{FO}(\theta; a, b)$ *is better than* $m_A(\theta; a, b)$ *in a neighborhood of the diagonal* $a = b$ *in the quarter plane* $(0, \infty) \times (0, \infty)$*.*

*Proof of Proposition 3.3.* It is sufficient to show the inequalities (VI.3.4) and (VI.3.5) with $0 <$

$a < b$. We have

$$\lim_{a\to 0} m_A(\theta, a, b) = (\theta \wedge (1-\theta))b \geq \lim_{a\to 0} m_{FO}(\theta, a, b) = \frac{\theta(1-\theta)}{2}b,$$
$$\lim_{a\to b} \frac{m_A(\theta, a, b)}{(a^{1/2}-b^{1/2})^2} = \theta \wedge (1-\theta) \leq \lim_{a\to b} \frac{m_{FO}(\theta, a, b)}{(a^{1/2}-b^{1/2})^2} = 2\theta(1-\theta).$$

Moreover, $m_A(\theta, a, b) = m_{FO}(\theta, a, b)$ is equivalent to the equation

$$((a/b)^{1/2}+1)^2 = \frac{2(\theta \wedge (1-\theta))}{\theta(1-\theta)}. \tag{VI.3.6}$$

Since the ratio of $a/b$ satisfies (VI.3.6) with given $\theta$ is uniquely determined, the inequalities (VI.3.4) and (VI.3.5) follow. Q.E.D.

To compare $M_A$ and $M_K$, we prepare Lambert's $W$ function, which is defined as the inverse function of $[-1, \infty) \ni x \mapsto xe^{1/x} \in [-1/e, \infty)$. For details, see [2].

**Proposition 3.5.** *Let* $0 \leq \theta \leq 1$ *and let*

$$t_1(\theta) = -\sqrt{2(\theta \wedge (1-\theta))}\; W\left(\frac{-1}{\sqrt{2(\theta \wedge (1-\theta))}} \exp\left(\frac{-1}{\sqrt{2(\theta \wedge (1-\theta))}}\right)\right),$$

*where* $t_1(0)$ *and* $t_1(1)$ *are understood to be*

$$\lim_{\theta\downarrow 0} t_1(\theta) = \lim_{\theta\uparrow 1} t_1(\theta) = 1.$$

*Then, the following inequalities hold for any* $a, b > 0$*:*

$$M_A(\theta; a, b) \leq M_K(\theta; a, b) \quad \text{if} \quad (a \wedge b) \leq t_1(\theta)(a \vee b), \tag{VI.3.7}$$
$$M_A(\theta; a, b) \geq M_K(\theta; a, b) \quad \text{if} \quad (a \wedge b) \geq t_1(\theta)(a \vee b). \tag{VI.3.8}$$

**Remark 3.6.** *Since* $0 \leq \theta \wedge (1-\theta) \leq 1/2$, $t_1(\theta)$ *satisfies* $0 \leq t_1(\theta) \leq 1$ *for* $0 \leq \theta \leq 1$. *In the proof below, we see that* $0 < t_1(\theta) < 1$ *if* $0 < \theta < 1$. *Proposition 3.5 shows that* $M_K(\theta; a, b)$ *is better than* $M_A(\theta)$ *in a neighborhood of the diagonal* $a = b$ *in the quarter plane* $(0, \infty) \times (0, \infty)$.

*Proof of Proposition 3.5.* Let $t > 0$ satisfy $t^2 = (a \wedge b)/(a \vee b)$. The magnitude correlation of $M_A(\theta, a, b)$ and $M_K(\theta; a, b)$ is coincide with that of

$$\sqrt{(a \vee b)^{-1} M_A(\theta, a, b)} = 1 - t$$

and

$$\sqrt{(a \vee b)^{-1} M_A(\theta, a, b)} = -\sqrt{2(\theta \wedge (1-\theta))}\log t.$$

Let $f(t) = 1 - t + \sqrt{2(\theta \wedge (1-\theta))}\log(t)$. We have $f(t_1(\theta)) = 0$, since

$$\begin{aligned}
&\frac{-t_1(\theta)}{\sqrt{2(\theta \wedge (1-\theta))}} \exp\left(\frac{-t_1(\theta)}{\sqrt{2(\theta \wedge (1-\theta))}}\right) \\
&= \frac{-1}{\sqrt{2(\theta \wedge (1-\theta))}} \exp\left(\frac{-1}{\sqrt{2(\theta \wedge (1-\theta))}}\right),
\end{aligned}$$

which is rewritten as

$$\exp\left(\frac{1-t_1(\theta)}{\sqrt{2(\theta \wedge (1-\theta))}}\right) = t_1(\theta)^{-1},$$

and moreover,

$$1 - t_1(\theta) = -\sqrt{2(\theta \wedge (1-\theta))}\log(t_1(\theta)).$$

In addition,

$$f'(t) = -1 + \sqrt{2(\theta \wedge (1-\theta))}/t.$$

Then, the inequalities (VI.3.7) and (VI.3.8) follow form the table below.

| $t$ | $0$ | $\cdots$ | $t_1(\theta)$ | $\cdots$ | $\sqrt{2(\theta \wedge (\theta - 1))}$ | $\cdots$ | $1$ |
|---|---|---|---|---|---|---|---|
| $f'(t)$ | $\infty$ | $+$ | $+$ | $+$ | $0$ | $-$ | $-$ |
| $f(t)$ | $-\infty$ | $\nearrow$ | $0$ | $\nearrow$ | $+$ | $\searrow$ | $0$ |

Q.E.D.

# 4 Dyadic Refinements of Multiplication Formulae and their Applications

In this section, we give dyadic refinements of the multiplication and dual multiplication formulae on the remainder function $R(\theta; a, b)$ and their applications. By the reciprocal formula $R(\theta; a, b) = R(1-\theta; b, a)$, it is important to describe the formation of the remainder function as $\theta \to 0$ and $\theta \to 1/2$ with the principal terms $2\theta R(1/2; a, b)$ and $2(1-\theta)R(1/2; a, b)$. For that purpose, we utilize dyadic decomposition.

**Proposition 4.1.** *Let $\theta$ satisfy $0 < \theta \le 2^{-n}$ with an integer $n \ge 1$. Then, the equality*

$$R(\theta, a, b) = \theta \sum_{j=1}^{n} 2^{j-1} b^{1-2^{1-j}} (a^{2^{-j}} - b^{2^{-j}})^2 + b^{1-2^{-n}} R(2^n\theta, a^{2^{-n}}, b^{2^{-n}}) \tag{VI.4.1}$$

*holds for all $a, b > 0$.*

***proof***. We apply Corollary 2.2 with $\sigma = 1/2$ to obtain

$$\begin{aligned}
&R(\theta; a, b) = \theta(a^{1/2} - b^{1/2}) + b^{1/2} R(2\theta; a^{1/2}, b^{1/2}),\\
&b^{1-2^{-j}} R(2^j\theta; a^{2^{-j}}, b^{2^{-j}})\\
&= b^{1-2^{-j}}\big(2^{j+1}\theta R(1/2; a^{2^{-j}}, b^{2^{-j}}) + b^{2^{-j-1}} R(2^{j+1}\theta; a^{2^{-j-1}}, b^{2^{-j-1}})\big)\\
&= 2^j\theta b^{1-2^{-j}}(a^{2^{-j}} - b^{2^{-j}})^2 + b^{1-2^{-j-1}} R(2^{j+1}\theta; a^{2^{-j-1}}, b^{2^{-j-1}})
\end{aligned}$$

for any $j$ with $1 \le j \le n$. Then, (VI.4.1) follows immediately. Q.E.D.

**Proposition 4.2.** *Let $\theta$ satisfy $(2^{m-1} - 1)/(2^m - 1) \le \theta \le 1/2$ with an integer $m \ge 1$. Then, the equality*

$$\begin{aligned}
R(\theta; a, b) &= (1-\theta)(a^{1/2} - b^{1/2})^2\\
&\quad - (1-2\theta)\sum_{j=1}^{m} 2^{j-1} a^\theta b^{(1-\theta)(1-2^{1-j})}\Big(a^{(1-\theta)2^{-j}} - b^{(1-\theta)2^{-j}}\Big)^2\\
&\quad - 2(1-\theta)a^\theta b^{(1-\theta)(1-2^{-m})} R\Big(2^m \cdot \frac{1/2-\theta}{1-\theta}; a^{(1-\theta)2^{-m}}, b^{(1-\theta)2^{-m}}\Big) \qquad \text{(VI.4.2)}
\end{aligned}$$

*holds for all $a, b > 0$.*

***proof***. We apply Corollary 2.5 with $\sigma = 1/2$ to obtain

$$R(\theta; a, b) = (1-\theta)(a^{1/2} - b^{1/2})^2 - 2(1-2\theta)a^\theta R\Big(\frac{1/2-\theta}{1-\theta}; a^{1-\theta}, b^{1-\theta}\Big). \qquad \text{(VI.4.3)}$$

Then, (VI.4.2) follows by applying Proposition 4.1 to the last term on the right hand side of (VI.4.3) with $0 \le (1/2-\theta)/(1-\theta) \le 2^{-m}$. Q.E.D.

**Corollary 4.3.** *Let $0 \le \theta \le 1/2$. Then, the inequalities*

$$\begin{aligned}
&\theta(a^{1/2} - b^{1/2})^2 + \big((2\theta \wedge (1-2\theta)\big)b^{1/2}(a^{1/4} - b^{1/4})^2\\
&\le R(\theta; a, b)\\
&\le (1-\theta)(a^{1/2} - b^{1/2})^2 - (1-2\theta)a^\theta\Big(a^{(1-\theta)/2} - b^{(1-\theta)/2}\Big)^2\\
&\qquad - 2\big(\theta \wedge (1-2\theta)\big)a^\theta b^{(1-\theta)/2}\Big(a^{(1-\theta)/4} - b^{(1-\theta)/4}\Big)^2 \qquad \text{(VI.4.4)}
\end{aligned}$$

*hold for all $a, b > 0$.*

**Corollary 4.4.** *Let $1/2 \le \theta \le 1$. Then, the inequalities*

$$\begin{aligned}
&(1-\theta)\big(a^{1/2} - b^{1/2}\big)^2 + \big((2(1-\theta)) \wedge (2\theta-1)\big)a^{1/2}\big(a^{1/4} - b^{1/4}\big)^2\\
&\le R(\theta; a, b)\\
&\le \theta\big(a^{1/2} - b^{1/2}\big)^2 - (2\theta-1)b^{1-\theta}\big(a^{\theta/2} - b^{\theta/2}\big)^2\\
&\qquad - 2\big((1-\theta) \wedge (2\theta-1)\big)a^{(1-\theta)/2}b^\theta\big(a^{\theta/4} - b^{\theta/4}\big)^2 \qquad \text{(VI.4.5)}
\end{aligned}$$

*hold for all $a, b > 0$.*

**Remark 4.5.** *Some of the lower bounds in Corollaries 4.3 and 4.4 may be found already in [7], Section 3.2*

**Remark 4.6.** *The inequalities* (VI.4.4) *and* (VI.4.5) *improve* (VI.2.5)*. The inequalities* (VI.2.5) *become an equality when* $\theta = 1/2$*, while* (VI.4.4) *become an equality when* $\theta = 0, 1/2$ *and* (VI.4.5) *become an equality when* $\theta = 1/2, 1$.

We are now in a position to apply the equalities above to Hölder type inequalities.

**Theorem 1.** *Let* $p$ *satisfy* $2 \le p < \infty$ *and let* $m$ *and* $n$ *be unique integers satisfying*

$$\begin{cases} 2^n \le p < 2^{n+1}, & n \ge 1, \\ (2^{m+1}-1)/(2^m-1) \le p < (2^m-1)/(2^{m-1}-1), & m \ge 1. \end{cases}$$

*Then, the equalities*

$$\begin{aligned}
&\|f\|_p\|g\|_{p'}\Bigg(1-\frac{1}{p'}\int_\Omega\left(\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\right)^2 d\mu \\
&+\left(\frac{1}{p'}-\frac{1}{p}\right)\sum_{j=1}^{m}2^{j-1}\int_\Omega\frac{|f|}{\|f\|_p}\frac{|g|^{1-2^{1-j}}}{\|g\|_{p'}^{1-2^{1-j}}}\left(\frac{|f|^{(p-1)2^{-j}}}{\|f\|_p^{(p-1)2^{-j}}}-\frac{|g|^{2^{-j}}}{\|g\|_{p'}^{2^{-j}}}\right)^2 d\mu \\
&+\frac{2}{p'}\int_\Omega\frac{|f|}{\|f\|_p}\frac{|g|^{(1-2^{-m})}}{\|g\|_{p'}^{(1-2^{-m})}}R\left(2^{m-1}\frac{p-2}{p-1},\frac{|f|^{(p-1)2^{-m}}}{\|f\|_p^{(p-1)2^{-m}}},\frac{|g|^{2^{-m}}}{\|g\|_{p'}^{2^{-m}}}\right)d\mu\Bigg) \\
&=\|fg\|_1 \\
&=\|f\|_p\|g\|_{p'}\Bigg(1-\frac{1}{p}\sum_{j=1}^{n}2^{j-1}\int_\Omega\frac{|g|^{p'(1-2^{1-j})}}{\|g\|_{p'}^{p'(1-2^{1-j})}}\left(\frac{|f|^{p2^{-j}}}{\|f\|_p^{p2^{-j}}}-\frac{|g|^{p'2^{-j}}}{\|g\|_{p'}^{p'2^{-j}}}\right)^2 d\mu \\
&-\int_\Omega\frac{|g|^{p'(1-2^{-n})}}{\|g\|_{p'}^{p'(1-2^{-n})}}R\left(\frac{2^n}{p},\ \frac{|f|^{p2^{-n}}}{\|f\|_p^{p2^{-n}}},\frac{|g|^{p'2^{-n}}}{\|g\|_{p'}^{p'2^{-n}}}\right)d\mu\Bigg) \qquad \text{(VI.4.6)}
\end{aligned}$$

*hold for all* $f \in L^p(\Omega,\mu)\backslash\{0\}$ *and* $g \in L^{p'}(\Omega,\mu)\backslash\{0\}$.

***proof***. The theorem follows from (VI.1.3) and Propositions 4.1 and 4.2 with $\theta = 1/p$, $a = |f|^p/\|f\|_p^p$, $b = |g|^{p'}/\|g\|_{p'}^{p'}$. Q.E.D.

**Corollary 4.7.** *Let $p, m, n$ be as in Theorem 1. Then, the inequalities*

$$\begin{aligned}
&\|f\|_p\|g\|_{p'}\Bigg(1-\frac{1}{p'}\int_\Omega\left(\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\right)^2 d\mu \\
&+\left(\frac{1}{p'}-\frac{1}{p}\right)\sum_{j=1}^{m}2^{j-1}\int_\Omega \frac{|f|}{\|f\|_p}\frac{|g|^{1-2^{1-j}}}{\|g\|_{p'}^{1-2^{1-j}}}\left(\frac{|f|^{(p-1)2^{-j}}}{\|f\|_p^{(p-1)2^{-j}}}-\frac{|g|^{2^{-j}}}{\|g\|_{p'}^{2^{-j}}}\right)^2 d\mu \\
&+\frac{2}{p'}\left(1-2^{m-1}\frac{p-2}{p-1}\right)\int_\Omega \frac{|f|}{\|f\|_p}\frac{|g|^{(1-2^{-m})}}{\|g\|_{p'}^{(1-2^{-m})}}\left(\frac{|f|^{(p-1)2^{-m-1}}}{\|f\|_p^{(p-1)2^{-m-1}}}-\frac{|g|^{2^{-m-1}}}{\|g\|_{p'}^{2^{-m-1}}}\right)^2 d\mu\Bigg) \\
&\le \|fg\|_1 \\
&\le \|f\|_p\|g\|_{p'}\Bigg(1-\frac{1}{p}\sum_{j=1}^{n}2^{j-1}\int_\Omega \frac{|g|^{p'(1-2^{1-j})}}{\|g\|_{p'}^{p'(1-2^{1-j})}}\left(\frac{|f|^{p2^{-j}}}{\|f\|_p^{p2^{-j}}}-\frac{|g|^{p'2^{-j}}}{\|g\|_{p'}^{p'2^{-j}}}\right)^2 d\mu \\
&\qquad -\frac{p-2^n}{p}\int_\Omega \frac{|g|^{p'(1-2^{-n})}}{\|g\|_{p'}^{p'(1-2^{-n})}}\left(\frac{|f|^{p2^{-n-1}}}{\|f\|_p^{p2^{-n-1}}}-\frac{|g|^{p'2^{-n-1}}}{\|g\|_{p'}^{p'2^{-n-1}}}\right)^2 d\mu\Bigg) \qquad \text{(VI.4.7)}
\end{aligned}$$

*hold for all $f \in L^p(\Omega,\mu)\backslash\{0\}$ and $g \in L^{p'}(\Omega,\mu)\backslash\{0\}$.*

***proof***. The required inequalities follows from Theorem 1 and Proposition 2.7. Q.E.D.

**Corollary 4.8.** *Let $p \ge 2^n$ with a positive integer $n$. Then, the inequalities*

$$\begin{aligned}
&\|f\|_p\|g\|_{p'}\Bigg(1-\frac{1}{p'}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\right\|_2^2 \\
&\qquad +\left(\frac{1}{p'}-\frac{1}{p}\right)\left\|\frac{|f|^{1/2}}{\|f\|_p^{1/2}}\left(\frac{|f|^{(p-1)/2}}{\|f\|_p^{(p-1)/2}}-\frac{|g|^{1/2}}{\|g\|_{p'}^{1/2}}\right)\right\|_2^2\Bigg) \\
&\le \|fg\|_1 \\
&\le \|f\|_p\|g\|_{p'}\Bigg(1-\frac{1}{p}\sum_{j=1}^{n}2^{j-1}\left\|\frac{|g|^{p'(1/2-2^{-j})}}{\|g\|_{p'}^{p'(1/2-2^{-j})}}\left(\frac{|f|^{p2^{-j}}}{\|f\|_p^{p2^{-j}}}-\frac{|g|^{p'2^{-j}}}{\|g\|_{p'}^{p'2^{-j}}}\right)\right\|_2^2 \\
&\qquad -\frac{p-2^n}{p}\left\|\frac{|g|^{p'(1/2-2^{-n-1})}}{\|g\|_{p'}^{p'(1/2-2^{-n-1})}}\left(\frac{|f|^{p2^{-n-1}}}{\|f\|_p^{p2^{-n-1}}}-\frac{|g|^{p'2^{-n-1}}}{\|g\|_{p'}^{p'2^{-n-1}}}\right)\right\|_2^2\Bigg) \qquad \text{(VI.4.8)}
\end{aligned}$$

*hold for all $f \in L^p(\Omega,\mu)\backslash\{0\}$ and $g \in L^{p'}(\Omega,\mu)\backslash\{0\}$.*

**Remark 4.9.** *In the case where $n = 1$ in Corollary 4.8, the coefficients of the upper and lower* *bounds of* $\|fg\|_1$ *are symmetric as follows:*

$$
\begin{aligned}
&\|f\|_p\|g\|_{p'}\left(1-\frac{1}{p'}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\right\|_2^2\right.\\
&\qquad\left.+\left(\frac{1}{p'}-\frac{1}{p}\right)\left\|\frac{|f|^{1/2}}{\|f\|_p^{1/2}}\left(\frac{|f|^{(p-1)/2}}{\|f\|_p^{(p-1)/2}}-\frac{|g|^{1/2}}{\|g\|_{p'}^{1/2}}\right)\right\|_2^2\right)\\
&\le \|fg\|_1\\
&\le \|f\|_p\|g\|_{p'}\left(1-\frac{1}{p}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\right\|_2^2\right.\\
&\qquad\left.+\left(\frac{1}{p}-\frac{1}{p'}\right)\left\|\frac{|g|^{p'/4}}{\|g\|_{p'}^{p'/4}}\left(\frac{|f|^{p/4}}{\|f\|_p^{p/4}}-\frac{|g|^{p'/4}}{\|g\|_{p'}^{p'/4}}\right)\right\|_2^2\right).
\end{aligned}
$$

**Remark 4.10.** *The inequalities* (VI.4.8) *improve Aldaz' stability version of the Hölder inequality [1]*

$$
\begin{aligned}
&\|f\|_p\|g\|_{p'}\left(1-\frac{1}{p'}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\right\|_2^2\right)\\
&\le \|fg\|_1 \le \|f\|_p\|g\|_{p'}\left(1-\frac{1}{p}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\right\|_2^2\right). \qquad \text{(VI.4.9)}
\end{aligned}
$$

*As Aldaz observed,* (VI.4.9) *become*

$$\|f\|_p\|g\|_{p'}\left(1-\frac{2}{p'}\right) \le \|fg\|_1 = 0 \le \|f\|_p\|g\|_{p'}\left(1-\frac{2}{p}\right)$$

*if* $\mathrm{supp}f \cap \mathrm{supp}g = \emptyset$. *In this respect, Corollary 4.8 is sharp since both sides of the inequalities* *in* (VI.4.8) *vanish as follows:*

$$
\begin{aligned}
&\|f\|_p\|g\|_{p'}\left(1-\frac{1}{p'}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\right\|_2^2\right.\\
&\qquad\left.+\left(\frac{1}{p'}-\frac{1}{p}\right)\left\|\frac{|f|^{1/2}}{\|f\|_p^{1/2}}\left(\frac{|f|^{(p-1)/2}}{\|f\|_p^{(p-1)/2}}-\frac{|g|^{1/2}}{\|g\|_{p'}^{1/2}}\right)\right\|_2^2\right)\\
&=\|f\|_p\|g\|_{p'}\left(1-\frac{2}{p'}+\frac{1}{p'}-\frac{1}{p}\right)=0,\\
&\|f\|_p\|g\|_{p'}\left(1-\frac{1}{p}\sum_{j=1}^{n}2^{j-1}\left\|\frac{|g|^{p'(1/2-2^{-j})}}{\|g\|_{p'}^{p'(1/2-2^{-j})}}\left(\frac{|f|^{p2^{-j}}}{\|f\|_p^{p2^{-j}}}-\frac{|g|^{p'2^{-j}}}{\|g\|_{p'}^{p'2^{-j}}}\right)\right\|_2^2\right.\\
&\qquad\left.-\frac{p-2^n}{p}\left\|\frac{|g|^{p'(1/2-2^{-n-1})}}{\|g\|_{p'}^{p'(1/2-2^{-n-1})}}\left(\frac{|f|^{p2^{-n-1}}}{\|f\|_p^{p2^{-n-1}}}-\frac{|g|^{p'2^{-n-1}}}{\|g\|_{p'}^{p'2^{-n-1}}}\right)\right\|_2^2\right)\\
&=\|f\|_p\|g\|_{p'}\left(1-\frac{2}{p}-\frac{1}{p}\sum_{j=2}^{n}2^{j-1}-\frac{p-2^n}{p}\right)=0.
\end{aligned}
$$

*In addition,* (VI.4.8) *coincides with the polarization identity*

$$
(|f|,|g|)=\|f\|_2\|g\|_2\left(1-\frac{1}{2}\left\|\frac{|f|}{\|f\|_2}-\frac{|g|}{\|g\|_2}\right\|_2^2\right)
$$

*when* $p=2$*, where* $(\cdot,\cdot)$ *is the standard* $L^2$ *inner product.*

## References


[1] J.M. Aldaz, "A stability version of Hölder's inequality," J. Math. Anal. Appl., **343**(2008), 842–852.

[2] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D. E. Knuth, "On the Lambert *W* function," Adv. Comput. Math., **5**(1996), 329–359.

[3] K. Fujiwara and T. Ozawa, "Exact remainder formula for the Young inequality and applications," International J. Math. Anal., **7**(2013), 2733 – 2735.

[4] S. Furuichi and N. Minculete, "Alternative reverse inequalities for Young's inequality," J. Math. Inequal., **5**(2011), 595–600.

[5] X. Gao, M. Gao, and X. Shang, "A refinement of Hölder's inequality and applications," JIPAM. J. Inequal. Pure Appl. Math., **8**(2007), Article 44, 9 pages.

[6] X.-L. Hu "An extension of Young's inequality and its application", Appl. Math. Comput., **219**(2013), 6393–6399.

[7] S. Kichenassamy, "Improving Hölder's inequality," Houston J. Math., **36**(2010), 303–312.

[8] J. Pečarić and V. Šimić, "A note on the Hölder inequality," JIPAM. J. Inequal. Pure Appl. Math., **7**(2006), Article 176, 3 pages.

**Part VII**

# Appendix D: Weighted $L^p$-boundedness of convolution type integral operators associated with bilinear estimates in the Sobolev spaces

**Abstract.** We study the boundedness of integral operators of convolution type in the Lebesgue spaces with weights. As a byproduct, we give a simple proof of the fact that the standard Sobolev space $H^s(\mathbb{R}^n)$ forms an algebra for $s > n/2$. Moreover, an optimality criterion is presented in the framework of weighted $L^p$-boundedness.

2000 Mathematics Subject Classification: 26D15, 46E35, 42B

Keywords: Pointwise multiplication, Sobolev spaces

## 1 Introduction

We study the boundedness of integral operators of convolution type in the Lebesgue space with weights. A special attention will be made on an optimality criterion with respect to the growth rate of weights.

To illustrate the problem, we revisit the standard property that the Sobolev space $H^s(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^2(\mathbb{R}^n)$ forms an algebra for $s > n/2$ from the point of view from the weighted $L^2(\mathbb{R}^n)$-boundedness of convolution. The corresponding bilinear estimate in the Sobolev space takes the form

$$\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s} \tag{VII.1.1}$$

with $s > n/2$, where

$$\|u\|_{H^s} = \|(1-\Delta)^{s/2}u\|_{L^2} = \|(1+|\xi|^2)^{s/2}\hat{u}\|_{L^2},$$
$$\hat{u}(\xi) = \mathfrak{F}u(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \exp(-ix\cdot\xi)u(x)dx,$$

and $\Delta$ is the Laplacian in $\mathbb{R}^n$. The bilinear estimate of this type was may be traced back at least to the paper by Saut and Temam [15]. There are many papers on further refinements and improvements on this subject as well as various applications to nonlinear partial differential equations. (see for instance [2–18] and references therein.)

One of the purpose in this part is to give a simple and elementary proof of (VII.1.1), which avoids paradifferential technique for instance.

In the Fourier representation, multiplication of functions is realized by convolution of the corresponding Fourier transformed functions:

$$\mathfrak{F}(uv)(\xi) = (2\pi)^{n/2}(\hat{u} * \hat{v})(\xi) = (2\pi)^{n/2}\int_{\mathbb{R}^n} \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta$$

and the estimate (VII.1.1) is equivalent to the bilinear estimate of the form

$$\|\omega(\hat{u} * \hat{v})\|_{L^2} \leq C\|\omega\hat{u}\|_{L^2}\|\omega\hat{v}\|_{L^2} \tag{VII.1.2}$$

with $\omega(\xi) = (1+|\xi|^2)^{s/2}$, which is also rewritten as

$$\left\|\omega\Big(\Big(\frac{\hat{u}}{\omega}\Big) * \Big(\frac{\hat{v}}{\omega}\Big)\Big)\right\|_{L^2} \leq C\|\hat{u}\|_{L^2}\|\hat{v}\|_{L^2}. \tag{VII.1.3}$$

By a duality argument, (VII.1.3) is equivalent to the trilinear estimate of the form

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \omega(\xi)\frac{1}{\omega(\xi-\eta)}\frac{1}{\omega(\eta)}\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(\xi)\,d\eta\,d\xi\right| \\ \leq C\|\hat{u}\|_{L^2}\|\hat{v}\|_{L^2}\|\hat{w}\|_{L^2}. \tag{VII.1.4}$$

By a simple change of variables, (VII.1.4) is equivalent to

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \omega(\xi+\eta)\frac{1}{\omega(\xi)}\frac{1}{\omega(\eta)}\hat{u}(\xi)\hat{v}(\eta)\hat{w}(\xi+\eta)\,d\eta\,d\xi\right| \\ \leq C\|\hat{u}\|_{L^2}\|\hat{v}\|_{L^2}\|\hat{w}\|_{L^2}. \tag{VII.1.5}$$

This gives a motivation to study the boundedness of the integrals of the form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w_0(x+y)w_1(x)w_2(y)f(x+y)g(x)h(y)\,dx\,dy \tag{VII.1.6}$$

with weight functions $w_0$, $w_1$, $w_2$, where $w_1$ and $w_2$ are supposedly the inverse weight of $w_0$.

The following theorem is basic in this direction.

**Theorem 1.** *Let $2 \leq p < \infty$ and let $w_0$, $w_1$, $w_2$ be nonnegative, continuous functions on $[0,\infty)$ satisfying*

$$M_1 \equiv \sup_{r>0} w_0^{\#}(2r)w_2(r)\|w_1(|\cdot|)\|_{L^p(B(r))} < \infty, \tag{VII.1.7}$$

$$M_2 \equiv \sup_{r>0} w_0^{\#}(2r)w_1(r)\|w_2(|\cdot|)\|_{L^p(B(r))} < \infty, \tag{VII.1.8}$$

*where*

$$w_0^{\#}(r) = \sup_{0\leq\rho\leq r} w_0(\rho),$$
$$B(r) = \{x \in \mathbb{R}^n; |x| \leq r\}.$$

*Then, the trilinear estimate*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w_0(|x+y|)\ w_1(|x|)\ w_2(|y|)\ |f(x+y)g(x)h(y)|\,dx\,dy \\ \leq (M_1+M_2)\|f\|_{L^p}\|g\|_{L^{p'}}\|h\|_{L^{p'}} \tag{VII.1.9}$$

*holds for all $f \in L^p(\mathbb{R}^n)$, $g$, $h \in L^{p'}(\mathbb{R}^n)$, where $p'$ is the dual exponent defined by $1/p+1/p' = 1$.*

***proof***. For $f \in L^p$ we define the translation by $y \in \mathbb{R}^n$ by $(\tau_y f)(x) = f(x+y)$. For $S \subset \mathbb{R}^n$ we denote by $\chi_S$ its characteristic function. Then, by the Hölder and Minkowski inequalities, we obtain

$$\begin{aligned}
&\iint_{|x|\le|y|} w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\,dx\,dy \\
&\le \iint w_0^{\#}(2|y|)\chi_{B(|y|)}(x)w_1(|x|)w_2(|y|)|\tau_y f(x)g(x)h(y)|\,dx\,dy \\
&\le \int w_0^{\#}(2|y|)\|\chi_{B(|y|)}w_1(|\cdot|)\|_{L^p}\|\tau_y f\cdot g\|_{L^{p'}}w_2(|y|)|h(y)|\,dy \\
&\le M_1\left(\int \|\tau_y f\cdot g\|_{L^{p'}}^p dy\right)^{1/p}\|h\|_{L^{p'}} \\
&\le M_1\left(\int\left(\int |f(x+y)|^p dy\right)^{p'/p}|g(x)|^{p'}dx\right)^{1/p'}\|h\|_{L^{p'}} \\
&= M_1\|f\|_{L^p}\|g\|_{L^{p'}}\|h\|_{L^{p'}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\iint_{|x|\ge|y|} w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\,dx\,dy \\
&\le \iint w_0^{\#}(2|x|)\chi_{B(|x|)}(y)w_1(|x|)w_2(|y|)|\tau_x f(y)g(x)h(y)|\,dx\,dy \\
&\le \int w_0^{\#}(2|x|)\|\chi_{B(|x|)}w_2(|\cdot|)\|_{L^p}\|\tau_x f\cdot h\|_{L^{p'}}w_1(|x|)|g(x)|\,dx \\
&\le M_2\left(\int \|\tau_x f\cdot h\|_{L^{p'}}^p dx\right)^{1/p}\|g\|_{L^{p'}} \\
&\le M_2\left(\int\left(\int |f(x+y)|^p dx\right)^{p'/p}|h(x)|^{p'}dy\right)^{1/p'}\|g\|_{L^{p'}} \\
&= M_2\|f\|_{L^p}\|g\|_{L^{p'}}\|h\|_{L^{p'}}.
\end{aligned}$$

Summing those inequalities, we have (VII.1.9). Q.E.D.

**Corollary 1.1.** *Let $2 \le p < \infty$ and let $w_0$, $w_1$, $w_2$ be nonnegative, continuous functions on $[0,\infty)$ satisfying*

$$M_1' = \sup_{r>0} w_0(2r)w_2(r)\|w_1(|\cdot|)\|_{L^p(B(r))} < \infty, \tag{VII.1.10}$$

$$M_2' = \sup_{r>0} w_0(2r)w_1(r)\|w_2(|\cdot|)\|_{L^p(B(r))} < \infty, \tag{VII.1.11}$$

*and the estimate*

$$w_0(r) \le C'w_0(R) \tag{VII.1.12}$$

*for any $r$ and $R$ with $0 \le r \le R$ with $C' \ge 1$ independent of $r$ and $R$. Then, the trilinear estimate*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w_0(|x+y|)\ w_1(|x|)\ w_2(|y|)\ |f(y+x)g(x)h(y)|\,dx\,dy$$
$$\le C'(M_1' + M_2')\|f\|_{L^p}\|g\|_{L^{p'}}\|h\|_{L^{p'}}$$

*holds for all $f \in L^p(\mathbb{R}^n)$, $g$, $h \in L^{p'}(\mathbb{R}^n)$.*

***proof***. By (VII.1.12), we have $w_0^{\#}(2r) \le C'w_0(2r)$ for any $r \ge 0$. Then, the corollary follows from Theorem 1 Q.E.D.

The bilinear estimate (VII.1.1) follows by choosing $p = 2$, $w_0(r) = (1+r^2)^{s/2}$, $w_1(r) = w_2(r) = (1+r^2)^{-s/2}$ with $s > n/2$, which ensures the required square integrability. A natural question then arises in connection with minimal growth rate at infinity in space for $w_0$, $1/w_1$, $1/w_2$. Weight functions of the form $w(r) = (1+r^2)^{n/2}(1+\log(1+r))^s$ with $s > 1/2$ may be the first candidate with $w_0 = w$, $w_1 = w_2 = 1/w$. This is not optimal since $w(r) = (1+r^2)^{n/2}(1+\log(1+r))^{1/2}\big(1+\log(1+\log(1+r))\big)^s$ with $s > 1/2$ has a slower growth with keeping the required square integrability.

To describe emerging extra logarithmic factors in such an iteration procedure, it is convenient to introduce the following set $\mathscr{F}$ consisting of positive, continuous functions $w$ on $[0,\infty)$ satisfying $1/w \in L^1_{\mathrm{loc}}(0,\infty)$ and the following assumptions (A1) and (A2):

(A1) For any $a \in \mathbb{R}$, there exists $C_a \ge 1$ such that for any $r$ and $R$ with $0 \le r \le R$, $w$ satisfies the inequality

$$w(r)\left(\int_0^r \frac{1}{w(\rho)}d\rho + 1\right)^a \le C_a w(R)\left(\int_0^R \frac{1}{w(R)}d\rho + 1\right)^a.$$

(A2) There exists $C > 0$ such that the inequality

$$w(2r) \le Cw(r)$$

holds for all $r > 0$.

**Example 1.** The function $w$ defined by $w(r) = 1 + r$ belongs to $\mathscr{F}$ with $C_a = 1$ for $a \ge 0$, $C_a = e^a(-a)^{-a}$ for $a < 0$, and $C = 2$.

**Example 2.** The function $w$ defined by $w(r) = (1+r)^s$ with $s > 1$ belongs to $\mathscr{F}$ with $C_a = 1$ for $a \ge -s$,

$$C_a = (-a)^{-a}(a+s-as)^{\frac{s+as-a}{s-1}} s^{\frac{-2s+a-as}{s-1}}$$

for $a < -s$, and $C = 2^s$.

**Example 3.** The function $w$ defined by $w(r) = (1+r^2)^{s/2}$ for $s \ge 1$ belong to $\mathscr{F}$ with $C_a = 1$ for $a \ge 0$,

$$C_a = s^a(-a)^{-a}r_{s,a}^a(1+r_{s,a}^2)^{-a+(a-1)s/2}$$

for $a < 0$, where $r_{s,a}$ is defined uniquely by

$$r_{s,a}(1+r_{s,a}^2)^{s/2-1}\left(\int_0^{r_{s,a}}(1+\rho^2)^{-s/2}d\rho+1\right)=\frac{|a|}{s}$$

and $C = 2^s$.

**Example 4.** Let $w(r) = 1 + \log(1 + r)$. Then,

$$w(r)\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-2} \leq w(r)\left(\int_0^r \frac{1}{1+\rho}d\rho+1\right)^{-2}=\frac{1}{w(r)} \to 0$$

as $r \to \infty$. This means $w \notin \mathscr{F}$.

**Example 5.** The function $w$ defined by $w(r) = (1+r)(1+\log(1+r))$ belongs to $\mathscr{F}$ with $C_a = 1$ for $a \geq 0$,

$$C_a = (-a)^{-a}(1+\tilde{r}_{s,a})^{-1}(1+\log(1+\tilde{r}_{s,a}))^{-1}(2+\log(\tilde{r}_{s,a}))^a$$

for $a < 0$, where $\tilde{r}_{s,a}$ is uniquely defined by

$$(2+\log(1+\tilde{r}_{s,a}))\Big(1+\log(1+\log(1+\tilde{r}_{s,a}))\Big)=|a|,$$

and $C = 2 + 2\log 2$.

**Remark 1.2.** *For $w \in \mathscr{F}$, we apply (A1) with $a = 0$ to obtain*

$$\begin{aligned}&\int_0^r \frac{1}{w(\rho)}d\rho\\ &\leq \int_0^{2r}\frac{1}{w(\rho)}d\rho = \int_0^r \frac{1}{w(\rho)}d\rho + \int_0^r \frac{1}{w(\rho+r)}d\rho\\ &\leq (1+C_0)\int_0^r \frac{1}{w(\rho)}d\rho. \end{aligned} \tag{VII.1.13}$$

**Theorem 2.** *Let $2 \leq p < \infty$ and let $w \in \mathscr{F}$. Let $w_0$, $w_1$, $w_2$ be defined by*

$$w_0(r) = (1+r)^{(n-1)/p}w(r)^{1/p}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-a},$$

$$w_1(r) = (1+r)^{-(n-1)/p}w(r)^{-1/p}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-b},$$

$$w_2(r) = (1+r)^{-(n-1)/p}w(r)^{-1/p}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-c}$$

*with $a, b, c \in \mathbb{R}$ satisfying either (i) or (ii):*

*(i)* $a+b+c \geq 1/p, \quad a+b > 0, \quad a+c > 0.$

*(ii)* $a+b+c > 1/p, \quad a+b \geq 0, \quad a+c \geq 0.$

*Then, there exists $C > 0$ such that the trilinear estimate*

$$\iint w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|dx\,dy \\ \leq C\|f\|_{L^p}\|g\|_{L^{p'}}\|h\|_{L^{p'}} \tag{VII.1.14}$$

*holds for all $f \in L^p(\mathbb{R}^n)$, $g, h \in L^p(\mathbb{R}^n)$.*

**Remark 1.3.** *In the case where $p = 2$ and $b = c = 0$, assumption (i) is equivalent to $a \geq 1/2$. In the case where $p = 2$ and $b = c > 0$, assumption (i) is equivalent to $a \geq 1/2 - 2b$ with $a > -b$. In the case where $p = 2$ and $-a = b = c$, assumption (i) breaks down and (ii) is equivalent to $-a = b = c > 1/2$.*

**Proof of Theorem 2.** We prove that $w_0$, $w_1$ and $w_2$ defined in the theorem satisfy the assumptions (VII.1.10)-(VII.1.12) in Corollary 1.1. Let $r$ and $R$ satisfy $0 \leq r \leq R$. By (A1),

$$w(r)\left(\int_0^r \frac{1}{w(\rho)}d\rho + 1\right)^{-ap} \leq C_{-ap}w(R)\left(\int_0^R \frac{1}{w(\rho)}d\rho + 1\right)^{-ap},$$

which yields

$$w_0(r) \leq C_{-ap}^{1/p}w_0(R). \tag{VII.1.15}$$

By (A2) and (VII.1.13),

$$\begin{aligned} w_0(2r) &= (1+2r)^{(n-1)/p}w(2r)^{1/p}\left(\int_0^{2r} \frac{1}{w(r)}d\rho + 1\right)^{-a} \\ &\leq 2^{(n-1)/p}(1+r)^{(n-1)/p}C^{1/p}w(r)^{1/p}(1+C_0)^{(-a)_+}\left(\int_0^r \frac{1}{w(\rho)}d\rho + 1\right)^{-a} \\ &= 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}w_0(r), \end{aligned} \tag{VII.1.16}$$

which yields

$$w_0(2r)w_1(r) \leq 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}\left(\int_0^r \frac{1}{w(r)}d\rho + 1\right)^{-a-b}. \tag{VII.1.17}$$

We estimate $w_2(|\cdot|)$ in $L^p(B(r))$ as

$$\|w_2(|\cdot|)\|_{L^p(B(r))} \leq \omega_{n-1}^{1/p}\left(\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma\right)^{-pc}d\rho\right)^{1/p}, \tag{VII.1.18}$$

where $\omega_{n-1}$ is the surface measure of the unit ball. To estimate the right hand side of (VII.1.18) and $M_1'$ of Corollary 1.1, we distinguish four cases:

(i) $c \leq 0$.

(ii) $0 < c < 1/p$.

(iii) $c = 1/p$.

(iv) $c > 1/p$.

(i) In the case where $c \le 0$, we estimate

$$\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\right)^{-pc} d\rho \le \int_0^r \frac{1}{w(\rho)}\left(\int_0^r \frac{1}{w(\sigma)}d\sigma + 1\right)^{-pc} d\rho$$
$$\le \left(\int_0^r \frac{1}{w(\rho)}d\rho + 1\right)^{1-pc}.$$

Then, $M_1'$ is estimated as follows:

$$M_1' \le \sup_{r>0} 2^{(n-1)/p} C^{1/p} (1 + C_0)^{(-a)_+} \left(\int_0^r \frac{1}{w(r)}d\rho + 1\right)^{1/p-a-b-c}$$
$$= 2^{(n-1)/p} C^{1/p} (1 + C_0)^{(-a)_+}.$$

(ii) In the case where $0 < c < 1/p$, we estimate

$$\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\right)^{-pc} d\rho = \frac{1}{1-pc}\left(\left(\int_0^r \frac{1}{w(\sigma)}d\sigma + 1\right)^{1-pc} - 1\right)$$
$$\le \frac{1}{1-pc}\left(\int_0^r \frac{1}{w(\sigma)}d\sigma + 1\right)^{1-pc}.$$

Then, $M_1'$ is estimated as follows:

$$M_1' \le \frac{1}{1-pc} 2^{(n-1)/p} C^{1/p} (1 + C_0)^{(-a)_+}.$$

(iii) In the case where $c = 1/p$, we estimate

$$\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\right)^{-1} d\rho = \log\left(1 + \int_0^r \frac{1}{w(\rho)}d\rho\right).$$

Since $a+b>0$, $M_1'$ is estimated as follows:

$$\begin{aligned}
M_1' &\le C^{1/p}(1+C_0)^{(-a)_+} \\
&\quad \cdot \sup_{r>0} 2^{(n-1)/p}\left(\int_0^r \frac{1}{w(r)}d\rho+1\right)^{-a-b} \log\left(1+\int_0^r \frac{1}{w(\rho)}d\rho\right) \\
&= 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+} \sup_{r\ge 1} r^{-a-b}\log r \\
&= 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+} \frac{1}{e(a+b)}.
\end{aligned}$$

(iv) In the case where $c>1/p$, we estimate

$$\begin{aligned}
&\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma+1\right)^{-pc} d\rho \\
&= \frac{1}{1-pc}\left(\left(\int_0^r \frac{1}{w(\sigma)}d\sigma+1\right)^{1-pc} d\rho-1\right) \\
&\le \frac{1}{pc-1}.
\end{aligned}$$

Since $a+b\ge 0$, $M_1'$ is estimated as follows:

$$M_1' \le \frac{1}{pc-1}2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}.$$

$M_2'$ is estimated similarly. Then, the estimate (VII.1.14) follows from Corollary 1.1. Q.E.D.

In a way similar to the proof of Theorem 2, we have the following theorem for $p=\infty$.

**Theorem 3.** *Let $w\in\mathscr{F}$. Let $w_0$, $w_1$, $w_2$ be defined by*

$$\begin{aligned}
w_0(r) &= \left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-a}, \\
w_1(r) &= \left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-b}, \\
w_2(r) &= \left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-c}
\end{aligned}$$

*with $a,b,c\in\mathbb{R}$ satisfying*

$$a+b+c_- \ge 0 \qquad \text{and} \qquad a+b_-+c\ge 0,$$

*where $b_-=\min(0,-b)$, $c=\min(0,-c)$.*
*Then, there exists $C>0$ such that the trilinear estimate*

$$\iint w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\ dx\ dy \le C\|f\|_\infty\|g\|_1\|h\|_1$$

*holds for all* $f \in L^\infty(\mathbb{R}^n)$, $g, h \in L^1(\mathbb{R}^n)$.

Theorem 2 shows the importance of the class $\mathscr{F}$ to the trilinear estimate such as (VII.1.9). Accordingly, below we study the class $\mathscr{F}$ in details. In Section 2, we study a basic property of $\mathscr{F}$. In Section 3, we introduce arbitrarily and infinitely iterates of logarithm in connection with $\mathscr{F}$. A part of the arguments in Sections 2 and 3 are essentially given by Ando, Horiuchi, and Nakai [1]. We revisit them in the present framework for definiteness. In Section 4, we study optimality of Theorem 1.2.

## 2 A Basic Property of $\mathscr{F}$

In this section we prove:

**Proposition 2.1.** *For* $w \in \mathscr{F}$ *and* $a \in \mathbb{R}$, *we define* $W_a$ *by*

$$W_a(r) = w(r)\left(\int_0^r \frac{1}{w(\rho)}d\rho + 1\right)^a, \qquad r \geq 0.$$

*Then,* $W_a \in \mathscr{F}$.

***proof***. By definition, we see that $W_a$ is a positive, continuous function on $[0, \infty)$ satisfying $1/W_a \in L^1_{\mathrm{loc}}(0, \infty)$. By (A2) and Remark 1.2,

$$W_a(2r) \leq Cw(r)\left(\int_0^{2r} \frac{1}{w(\rho)}d\rho + 1\right)^a \leq C(C_0+1)^{a_+}W_a(r),$$

where $a_+ = \max(a, 0)$. It remains to prove that $W_a$ satisfies (A1); For any $a, b \in \mathbb{R}$, there exists $C_{a,b}$ such that for any $r$ and $R$ with $0 \leq r \leq R$,

$$W_a(r)\left(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\right)^b \leq C_{a,b}W_a(R)\left(\int_0^R \frac{1}{W_a(\rho)}d\rho + 1\right)^b$$

holds. Let $0 \leq r \leq R$. We note that (A1) property of $w$ is equivalent to $W_a(r) \leq C_a W_a(R)$. We distinguish three cases:

(i) $b \geq 0$. (ii) $b < 0$, $a \geq 0$. (iii) $b < 0$, $a < 0$.

(i) In the case where $b \geq 0$, we estimate

$$\begin{aligned} W_a(r)\left(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\right)^b &\leq C_a W_a(R)\left(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\right)^b \\ &\leq C_a W_a(R)\left(\int_0^R \frac{1}{W_a(\rho)}d\rho + 1\right)^b, \end{aligned}$$

as required.

(ii) In the case where $b < 0$ and $a \geq 0$, we first notice that

$$
\begin{aligned}
&\frac{1}{W_a(R)}\left(\int_0^R \frac{1}{W_a(\rho)}d\rho + 1\right)^{|b|} \\
&= \frac{1}{W_a(R)}\left(\int_0^r \frac{1}{W_a(\rho)}d\rho + \int_r^R \frac{1}{W_a(\rho)}d\rho + 1\right)^{|b|} \\
&\leq \frac{2^{(|b|-1)_+}}{W_a(R)}\left(\left(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\right)^{|b|} + \left(\int_r^R \frac{1}{W_a(\rho)}d\rho\right)^{|b|}\right) \\
&\leq \frac{C_a 2^{(|b|-1)_+}}{W_a(r)}\left(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\right)^{|b|} + \frac{C_a 2^{(|b|-1)_+}}{W_a(r)}\left(\int_r^R \frac{1}{W_a(\rho)}d\rho\right)^{|b|}. \qquad \text{(VII.2.1)}
\end{aligned}
$$

To estimate the second term on the right hand side of the last inequality of (VII.2.1), we remark that

$$
\begin{aligned}
\int_r^R \frac{1}{W_a(\rho)}d\rho &= \int_r^R \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\right)^{-a} d\rho \\
&\leq \int_r^R \frac{1}{w(\rho)}\left(\int_0^r \frac{1}{w(\sigma)}d\sigma + 1\right)^{-a} d\rho \\
&\leq \left(\int_0^r \frac{1}{w(\sigma)}d\sigma + 1\right)^{-a} \int_0^R \frac{1}{w(\rho)}d\rho
\end{aligned}
$$

and

$$
\frac{1}{W_a(R)} = \frac{1}{w(R)}\left(\int_0^R \frac{1}{w(\rho)}d\rho + 1\right)^{-a} \leq \frac{1}{w(R)}\left(\int_0^r \frac{1}{w(\rho)}d\rho + 1\right)^{-a}.
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{W_a(R)}\left(\int_r^R \frac{1}{W_a(\rho)}d\rho\right)^{|b|}\\
&\le \frac{1}{w(R)}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-a-a|b|}\left(\int_0^R \frac{1}{w(\rho)}d\rho\right)^{|b|}\\
&\le \left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-a-a|b|}\frac{1}{w(R)}\left(\int_0^R \frac{1}{w(\rho)}d\rho+1\right)^{|b|}\\
&\le \left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-a-a|b|}\cdot C_b\frac{1}{w(r)}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{|b|}\\
&\le C_b\frac{1}{w(r)\Big(\int_0^r \frac{1}{w(\rho)}d\rho+1\Big)^a}\cdot\left(\int_0^r \frac{1}{w(\rho)\Big(\int_0^r \frac{1}{w(\sigma)}d\sigma+1\Big)^a}d\rho+1\right)^{|b|}\\
&\le C_b\frac{1}{W_a(r)}\left(\int_0^r \frac{1}{W_a(\rho)}d\rho+1\right)^{|b|}. \qquad \text{(VII.2.2)}
\end{aligned}
$$

Combining (VII.2.1) and (VII.2.2) and taking the inverse of the resulting inequality, we find that $W_a$ satisfies (A1).

(iii) In the case where $b<0$ and $a<0$, we use the equality

$$\int_0^r \frac{1}{W_a(\rho)}d\rho+1=\frac{1}{|a|+1}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{|a|+1}+\frac{|a|}{|a|+1}$$

to estimate

$$
\begin{aligned}
&W_a(r)\left(\int_0^r \frac{1}{W_a(\rho)}d\rho+1\right)^{b}\\
&\le \frac{1}{(|a|+1)^b}w(r)\left(\int_0^r \frac{1}{w(\rho)}+1\right)^{a+(|a|+1)b}\\
&\le (|a|+1)^{|b|}C_{a+b-ab}w(R)\left(\int_0^R \frac{1}{w(\rho)}d\rho+1\right)^{a+b-ab}\\
&= (|a|+1)^{|b|}C_{a+b-ab}W_a(R)\left(\int_0^R \frac{1}{w(\rho)}d\rho+1\right)^{(|a|+1)b}\\
&\le (|a|+1)^{|b|}C_{a+b-ab}W_a(R)\left(\int_0^R \frac{1}{W_a(\rho)}d\rho+1\right)^{b},
\end{aligned}
$$

as required. Q.E.D.

## 3 Infinitely Iterated Logarithm

In this section, we introduce arbitrarily and infinitely iterated logarithm functions in connection with class $\mathscr{F}$. The definition is different from that of [1] in the sense that convergence factors are introduced in terms of the parameter $\theta \in (0,1]$.

**Definition 3.1.** *Let $0 < \theta \leq 1$. For nonnegative integers $n$, the following functions $l_{\theta,n} : [0,\infty) \to \mathbb{R}$ are defined successively by:*

$$l_{\theta,0}(r) = 1 + r,$$
$$l_{\theta,k}(r) = 1 + \theta \log l_{\theta,k-1}(r), \quad k \geq 1.$$

*Moreover, we define $L_{\theta,k} : [0,\infty) \to \mathbb{R}$ by*

$$L_{\theta,k}(r) = \prod_{j=0}^{k} l_{\theta,j}(r).$$

**Remark 3.2.** *For any $k \geq 0$, $l_{\theta,k}(0) = L_{\theta,k}(0) = 1$. Moreover, $l_{\theta,k}(r) \geq 1$ and $L_{\theta,k}(r) \geq 1$ for all $r \geq 0$ since $l_{\theta,k}$ and $L_{\theta,k}$ are increasing functions. Explicitly, the derivative $l'_{\theta,k}$ is given by*

$$l'_{\theta,k}(r) = \theta^k \cdot \frac{1}{L_{\theta,k-1}(r)}, \qquad r \geq 0.$$

*By a successive use of the elementary inequality $\log(1+r) \leq r$ for $r \geq -1$,*

$$0 \leq \log l_{\theta,k}(r) \leq \theta \log l_{\theta,k-1}(r) \leq \cdots \leq \theta^k \log l_{\theta,0}(r), \quad r \geq 0.$$

*This implies that for any $\theta$ with $0 < \theta < 1$, the series $\sum_{k=0}^{\infty} \log l_{\theta,k}(r)$ converges with estimates*

$$0 \leq \sum_{k=0}^{\infty} \log l_{\theta,k}(r) \leq \frac{1}{1-\theta} \log l_{\theta,0}(r), \quad r \geq 0.$$

**Definition 3.3.** *For any $\theta$ with $0 < \theta \leq 1$, $L_\theta$ is defined by*

$$L_\theta(r) = \prod_{k=0}^{\infty} l_{\theta,k}(r), \qquad r \geq 0.$$

**Remark 3.4.** *By Remark 3.2, if $0 < \theta < 1$, then $L_\theta$ converges with estimates*

$$1 \leq L_\theta(r) \leq (1+r)^{1/(1-\theta)}, \quad r \geq 0.$$

*If $\theta = 1$ and $r > 0$, we prove that $L_1(r) = \infty$ by contradiction. Assume that $L_1(r) < \infty$. Then,* *for any $k$ we have*

$$\log L_1(r) \geq \log L_{1,k}(r) = \int_0^r \frac{d}{d\rho}\left(\sum_{j=0}^{k} \log l_{1,j}(\rho)\right) d\rho$$
$$= \int_0^r \sum_{j=0}^{k} \frac{1}{L_{1,j}(\rho)} d\rho$$
$$\geq \int_0^r \sum_{j=0}^{k} \frac{1}{L_{1,k}(r)} d\rho = r \sum_{j=0}^{k} \frac{1}{L_{1,k}(r)} \geq \frac{(k+1)r}{L_1(r)},$$

*which yields a contradiction for $k$ sufficiently large.*

The main theorem in this section now reads:

**Theorem 4.** *For any $\theta$ with $0 < \theta < 1$, $L_\theta \in \mathscr{F}$. Moreover,*

$$\int_0^\infty \frac{1}{L_\theta(r)} dr = \infty.$$

To prove Theorem 4, we introduce some preliminary propositions. From now on, $\theta$ denotes a real number with $0 < \theta < 1$ without particular comments.

**Lemma 3.5.** *For any $a \in \mathbb{R}$, there exists $C_{\theta,a} \geq 1$ such that for any $r$ and $R$ with $0 \leq r \leq R$,*

$$(1+r)\left(\int_0^r \frac{1}{L_\theta(\rho)} d\rho + 1\right)^a \leq C_{\theta,a}(1+R)\left(\int_0^R \frac{1}{L_\theta(\rho)} d\rho + 1\right)^a \tag{VII.3.1}$$

*holds.*

***proof***. For $a \geq 0$, (VII.3.1) holds with $C_a = 1$ by monotonicity. Let $a < 0$ and let $m_\theta$ be defined by

$$m_\theta(r) = \int_0^r \frac{1}{L_\theta(\rho)} d\rho + 1.$$

Then,

$$m'_\theta(R) = \frac{1}{L_\theta(R)} \leq \frac{m_\theta(r)}{l_{\theta,1}(R) l_{\theta,0}(R)} \leq \frac{m_\theta(r)}{\theta l_{\theta,1}(r)} l'_{\theta,1}(R). \tag{VII.3.2}$$

By (VII.3.2), we have

$$
\begin{aligned}
m_\theta(R) &= m_\theta(r) + \int_r^R m'_\theta(\rho)d\rho \\
&\le m_\theta(r) + \frac{m_\theta(r)}{\theta l_{\theta,1}(r)} \int_r^R l'_{\theta,1}(\rho)d\rho \\
&= m_\theta(r) + \frac{m_\theta(r)}{\theta l_{\theta,1}(r)} \Big( l_{\theta,1}(R) - l_{\theta,1}(r) \Big) \\
&\le \frac{m_\theta(r)}{\theta l_{\theta,1}(r)} l_{\theta,1}(R). \qquad \text{(VII.3.3)}
\end{aligned}
$$

By Remark 3.2 and (VII.3.3), we obtain

$$
\begin{aligned}
(1+r)m_\theta(r)^a &= \left( \frac{m_\theta(r)}{l_{\theta,1}(r)} \right)^a (1+r)\Big( l_{\theta,1}(r) \Big)^a \\
&\le C \left( \frac{m_\theta(r)}{l_{\theta,1}(r)} \right)^a (1+R)\Big( l_{\theta,1}(R) \Big)^a \\
&\le C\theta^a (1+R) m_\theta(R)^a
\end{aligned}
$$

with some constant $C$, as required. Q.E.D.

**Lemma 3.6.** *For any* $r, s \ge 0$,

$$L_\theta(l_{\theta,0}(s)r) \le L_\theta(s)L_\theta(r). \qquad \text{(VII.3.4)}$$

***proof***. It is sufficient to prove that

$$l_{\theta,k}(l_{\theta,0}(s)r) \le l_{\theta,k}(s)l_{\theta,k}(r) \qquad (VII.3.5)_k$$

by induction on $k \ge 0$. For $k = 0$,

$$l_{\theta,0}(l_{\theta,0}(s)r) = 1 + l_{\theta,0}(s)r = 1 + (1+s)r \le (1+s)(1+r) = l_{\theta,0}(s)l_{\theta,0}(r)$$

Let $k \ge 1$ and assume $(\text{VII.3.5})_{k-1}$. Then,

$$
\begin{aligned}
l_{\theta,k}(l_{\theta,0}(s)r) &= 1 + \theta \log \Big( l_{\theta,k-1}(l_{\theta,0}(s)r) \Big) \\
&\le 1 + \theta \log \Big( l_{\theta,k-1}(s) l_{\theta,k-1}(r) \Big) \\
&\le \Big( 1 + \theta \log l_{\theta,k-1}(s) \Big) \Big( 1 + \theta \log l_{\theta,k-1}(r) \Big) \\
&\le l_{\theta,k}(s) l_{\theta,k}(r),
\end{aligned}
$$

which completes the induction argument. Q.E.D.

**Lemma 3.7.** *For any nonnegative integers $k$ and $j$, $l_{\theta,k+j}$ is represented by $l_{\theta,k}$ and $l_{\theta,j}$ as*

$$l_{\theta,k+j}(r) = l_{\theta,j}\Big(l_{\theta,k}(r) - 1\Big) \tag{VII.3.6}$$

*for all $r \geq 0$.*

***proof***. We prove (VII.3.6) by induction on $j$. For $j = 0$, we have

$$l_{\theta,k}(r) = l_{\theta,0}\Big(l_{\theta,k}(r) - 1\Big)$$

for all $k \geq 0$ by definition. Let $j \geq 1$ and assume that

$$l_{\theta,k+j-1}(r) = l_{\theta,j-1}\Big(l_{\theta,k}(r) - 1\Big)$$

holds for all $k \geq 0$ and $r \geq 0$. Then,

$$\begin{aligned} l_{\theta,k+j}(r) &= 1 + \theta \log\Big(l_{\theta,k+j-1}(r)\Big) \\ &= 1 + \theta \log\Big(l_{\theta,j-1}(l_{\theta,k}(r) - 1)\Big) \\ &= l_{\theta,j}\Big(l_{\theta,k}(r) - 1\Big) \end{aligned}$$

for all $k \geq 0$ and $r \geq 0$. This completes the induction argument. Q.E.D.

**Proof of Theorem 4.** Let $r$, $R$ satisfy $0 \leq r \leq R$. Then, by Lemma 3.5,

$$\begin{aligned} L_\theta(r)\left(\int_0^r \frac{1}{L_\theta(\rho)} d\rho + 1\right)^a &\leq (1 + r)\left(\prod_{k=1}^{\infty} l_{\theta,k}(R)\right)\left(\int_0^r \frac{1}{L_\theta(\rho)} d\rho + 1\right)^a \\ &\leq C_{\theta,a} L_\theta(R)\left(\int_0^R \frac{1}{L_\theta(\rho)} d\rho + 1\right)^a. \end{aligned}$$

Moreover, since $l_{\theta,0}(1) = 2$, we apply (VII.3.4) with $s = 1$ to obtain

$$L_\theta(2r) \leq L_\theta(1) L_\theta(r).$$

Therefore, $L_\theta \in \mathscr{F}$. We prove (VII.3.4). It suffices to prove that there exists a sequence $\{r_k; k \geq 0\}$ of positive numbers such that

$$\int_0^{r_k} \frac{1}{L_\theta(\rho)} d\rho \to \infty$$

as $k \to \infty$. Let $r_0 = 1$. Then, for any $k \geq 1$ there exists a unique $r_k > 0$ such that $l_{\theta,k}(r_k) = l_{\theta,0}(r_0) = 2$, since $l_{\theta,k}$ is an increasing function with $l_{\theta,k}(0) = 1$ and $\lim_{r\to\infty} l_{\theta,k}(r) = \infty$. Let

$0 \le \rho \le r_k$. By Lemma 3.7,

$$\begin{aligned}
L_\theta(\rho) &= L_{\theta,k-1}(\rho) \prod_{j=0}^{\infty} l_{\theta,k+j}(\rho) \\
&\le L_{\theta,k-1}(\rho) \prod_{j=0}^{\infty} l_{\theta,k+j}(r_k) \\
&= L_{\theta,k-1}(\rho) \prod_{j=0}^{\infty} l_{\theta,j}\Big(l_{\theta,k}(r_k) - 1\Big) \\
&= L_{\theta,k-1}(\rho)\, L_\theta\Big(l_{\theta,k}(r_k) - 1\Big) \\
&= L_{\theta,k-1}(\rho)\, L_\theta\Big(l_{\theta,0}(r_0) - 1\Big) \\
&= L_{\theta,k-1}(\rho)\, L_\theta(1). 
\end{aligned} \tag{VII.3.7}$$

By (VII.3.6) and (VII.3.7),

$$\begin{aligned}
\int_0^{r_k} \frac{1}{L_\theta(\rho)} d\rho \ge \frac{1}{L_\theta(1)} \int_0^{r_k} \frac{1}{L_{\theta,k-1}(\rho)} d\rho
&= \frac{1}{L_\theta(1)} \frac{1}{\theta^k} \Big(l_{\theta,k}(r_k) - 1\Big) \\
&= \frac{1}{L_\theta(1)} \frac{1}{\theta^k} \to \infty
\end{aligned}$$

as $k \to \infty$, as required. Q.E.D.

# 4 Optimality of Theorems 2 and 3.

In this section, we consider optimality of Theorems 2 and 3. To this end, we divide weight functions $w \in \mathscr{F}$ into two cases:

$$\text{I}: \int_0^\infty \frac{1}{w(r)} dr < \infty. \qquad \text{II}: \int_0^\infty \frac{1}{w(r)} dr = \infty.$$

**Theorem 5.** *Let $2 \le p < \infty$ and let $w \in \mathscr{F}$. Let $w_0$, $w_1$, $w_2$ be as in Theorem 2 with $a$, $b$, $c \in \mathbb{R}$.*
*(1) In the case I, the trilinear estimate in Theorem 2 holds for any $a$, $b$, $c \in \mathbb{R}$.*

*(2) In the case II, let $a$, $b$, $c$ satisfy one of the conditions (iii), (iv), (v), (vi):*

*(iii) $a + b + c < 1/p$.*

*(iv) $a + b < 0$.*

*(v) $a + c < 0$.*

*(vi) $a + b + c = 1/p$ and "$a + b = 0$ or $a + c = 0$".*

*Then, the trilinear estimate in Theorem 2 fails for some $f \in L^p(\mathbb{R}^n)$ and $g$, $h \in L^{p'}(\mathbb{R}^n)$.*

**Remark 4.1.** *The conditions (iii), (iv), (v), and (vi) in Theorem 5 consist of the negation of the condition "(i) or (ii)" in Theorem 2.*

***proof***. In the case I, we easily see the trilinear estimate holds with any $a$, $b$, and $c$. To give a counter example for the trilinear estimate in the case II, we divide the proof into three cases:

(i) $a+b+c<1/p$.

(ii) $a+b<0$ or $a+c<0$.

(iii) $a+b+c=1/p$ and "$a+b=0$ or $a+c=0$".

(i) In the case where $a+b+c<1/p$, let $\delta>0$ satisfy $\delta \neq 1/p-c$ and let

$$f(x)=(1+|x|)^{-(n-1)/p}w(|x|)^{-1/p}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p-\delta},$$

$$g(x)=h(x)=(1+|x|)^{-(n-1)/p'}w(|x|)^{-1/p'}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'-\delta}.$$

Then, $f \in L^p(\mathbb{R}^n)$ and $g,h \in L^{p'}(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$ with $|x| \geq 2$,

$$\begin{aligned}
&\int_{1\leq|y|\leq|x|/2} w_0(|x+y|)f(x+y)w_2(|y|)h(y)dy \\
&=\int_{1\leq|y|\leq|x|/2}\left(\int_0^{|x+y|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p-a-\delta} \\
&\quad\cdot(1+|y|)^{-(n-1)}\frac{1}{w(|y|)}\left(\int_0^{|y|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'-c-\delta} dy. \qquad \text{(VII.4.1)}
\end{aligned}$$

By (A1), if $1/p+a+\delta \geq 0$, then for any $y \in \mathbb{R}^n$ with $0 \leq |y| \leq |x|/2$,

$$\begin{aligned}
&\left(\int_0^{|x+y|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p-a-\delta} \\
&\geq\left(\int_0^{3|x|/2}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p-a-\delta} \\
&=\left(\frac{3}{2}\int_0^{|x|}\frac{1}{w(3\rho/2)}d\rho+1\right)^{-1/p-a-\delta} \\
&\geq\left(\frac{3C_0}{2}\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p-a-\delta} \\
&\geq\left(\frac{3C_0}{2}+1\right)^{-1/p-a-\delta}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p-a-\delta}. \qquad \text{(VII.4.2)}
\end{aligned}$$

Similarly, if $1/p+a+\delta<0$, then for any $y\in\mathbb{R}^n$ with $0\le|y|\le|x|/2$,

$$\begin{aligned}
&\left(\int_0^{|x+y|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p-a-\delta}\\
&\ge\left(\int_0^{|x|/2}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p-a-\delta}\\
&=\left(\frac{1}{2}\int_0^{|x|}\frac{1}{w(\rho/2)}d\rho+1\right)^{-1/p-a-\delta}\\
&\ge\left(\frac{1}{2C_0}\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p-a-\delta}\\
&\ge(2C_0)^{1/p+a+\delta}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p-a-\delta}. \qquad \text{(VII.4.3)}
\end{aligned}$$

In addition, if $1/p-c-\delta\ge0$, then for any $x\in\mathbb{R}^n$ with $|x|\ge4$,

$$\begin{aligned}
&\int_{1\le|y|\le|x|/2}(1+|y|)^{-(n-1)}\frac{1}{w(|y|)}\left(\int_0^{|y|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'-c-\delta}dy\\
&=\omega_{n-1}\int_1^{|x|/2}\left(\frac{r}{1+r}\right)^{n-1}\frac{1}{w(r)}\left(\int_0^{r}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'-c-\delta}dr\\
&\ge\frac{2^{1-n}\omega_{n-1}}{1/p-c-\delta}\left(\left(\int_0^{|x|/2}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}-\left(\int_0^{1}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}\right)\\
&\ge\frac{2^{1-n}\omega_{n-1}}{1/p-c-\delta}\left(1-\left(\int_0^{2}\frac{1}{w(r)}dr+1\right)^{-1/p+c+\delta}\right)\\
&\quad\cdot\left(\int_0^{|x|/2}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}\\
&=\frac{2^{1-n}\omega_{n-1}}{1/p-c-\delta}\left(1-\left(\int_0^{2}\frac{1}{w(r)}dr+1\right)^{-1/p+c+\delta}\right)\\
&\quad\cdot\left(\frac{1}{2}\int_0^{|x|}\frac{1}{w(r/2)}dr+1\right)^{1/p-c-\delta}\\
&\ge\frac{2^{1/p'-c-\delta-n}\omega_{n-1}}{(1/p-c-\delta)C_0^{1/p-c-\delta}}\left(1-\left(\int_0^{2}\frac{1}{w(r)}dr+1\right)^{-1/p+c+\delta}\right)\\
&\quad\cdot\left(\int_0^{|x|}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}. \qquad \text{(VII.4.4)}
\end{aligned}$$

If $1/p - c - \delta < 0$, then

$$\begin{aligned}
&\int_{1\le|y|\le|x|/2} (1+|y|)^{-(n-1)} \frac{1}{w(|y|)} \left( \int_0^{|y|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p'-c-\delta} dy \\
&= \frac{2^{1-n}\omega_{n-1}}{1/p-c-\delta} \left( \left( \int_0^1 \frac{1}{w(r)} dr + 1 \right)^{1/p-c-\delta} - \left( \int_0^{|x|/2} \frac{1}{w(r)} dr + 1 \right)^{1/p-c-\delta} \right) \\
&\ge \frac{2^{1-n}\omega_n}{1/p-c-\delta} \left( \left( \int_0^1 \frac{1}{w(r)} dr + 1 \right)^{1/p-c-\delta} - \left( \int_0^2 \frac{1}{w(r)} dr + 1 \right)^{1/p-c-\delta} \right) \\
&\quad \cdot \left( \int_0^{|x|} \frac{1}{w(r)} dr + 1 \right)^{1/p-c-\delta}. \qquad \text{(VII.4.5)}
\end{aligned}$$

By (VII.4.1), (VII.4.2), (VII.4.3), (VII.4.5), and (VII.4.5), there exists a positive constant $C$ such that for any $x \in \mathbb{R}^n$ with $|x| \ge 4$,

$$\begin{aligned}
&\int_{1\le|y|\le|x|/2} w_0(|x+y|) f(x+y) w_2(|y|) h(y) dy \\
&\ge C \left( \int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-a-c-2\delta}. \qquad \text{(VII.4.6)}
\end{aligned}$$

Finally by (VII.4.6), we have

$$\begin{aligned}
&\iint w_0(|x+y|) w_1(|x|) w_2(|y|) |f(x+y) g(x) h(y)| \, dx \, dy \\
&\ge C \int_{|x|\ge 4} (|x|+1)^{-(n-1)} \frac{1}{w(|x|)} \left( \int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p'-a-b-c-3\delta} dx \\
&\ge C\omega_n \left( \frac{4}{5} \right)^{n-1} \int_4^\infty \frac{1}{w(r)} \left( \int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p'-a-b-c-3\delta} dr \\
&\ge C\omega_n \left( \frac{4}{5} \right)^{n-1} \left( \log \left( \int_0^\infty \frac{1}{w(\rho)} d\rho + 1 \right) - \log \left( \int_0^4 \frac{1}{w(\rho)} d\rho + 1 \right) \right) \\
&= \infty
\end{aligned}$$

with $\delta \le (1/p - a - b - c)/3$.

(ii) In the case where $a + b < 0$ or $a + c < 0$, by symmetry, it is sufficient to give a counter example only in the case where $a+b < 0$. Let $f$ and $g$ be as in the case (iii) with $\delta \le -(a+b)/2$ and $a + 1/p + \delta \ne 1$. Let

$$h(x) = \chi_{B(1)}(x) \frac{1}{w_2(|x|)}.$$

Then, by (VII.1.15), (VII.4.4), and (VII.4.5),

$$
\begin{aligned}
&\iint w_0(|x+y|)w_1(|x|)w_2(|y|)f(x+y)g(x)h(y)\ dy\,dx \\
&\geq \int_{|x|\geq 2}\int_{|y|\leq 1}\left(\int_0^{|x+y|}\frac{1}{w(\rho)}d\rho+1\right)^{-a-1/p-\delta}dy \\
&\qquad\qquad \cdot (1+|x|)^{n-1}\frac{1}{w(|x|)}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-b-1/p'-\delta}dx \\
&\geq C\int_2^\infty \frac{1}{w(r)}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-a-b-1-2\delta}dr \\
&\geq C\int_2^\infty \frac{1}{w(r)}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-1}dr \\
&\geq C\left(\log\left(\int_0^\infty \frac{1}{w(\rho)}d\rho+1\right)-\log\left(\int_0^2 \frac{1}{w(\rho)}d\rho+1\right)\right) \\
&= \infty
\end{aligned}
$$

with some positive constant $C$, as required.

(iii) In the case where $a+b+c=1/p$ and $a+b=0$ or $a+c=0$, by symmetry, it is sufficient to give a counter example in the case where $a+b=0$. Let

$$
\begin{aligned}
J(r) &= \int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma+1\right)^{-1}d\rho+1, \\
f(x) &= (1+|x|)^{-(n-1)/p}w(|x|)^{-1/p}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p}J(|x|)^{-1/p-\delta}, \\
g(x) &= h(x) \\
&= (1+|x|)^{-(n-1)/p'}w(|x|)^{-1/p'}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'}J(|x|)^{-1/p'-\delta}
\end{aligned}
$$

for $\delta>0$. By (A1),

$$
\begin{aligned}
J(2r) &= \int_0^{2r}\frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma+1\right)^{-1}d\rho+1 \\
&\leq \int_0^{r}\frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma+1\right)^{-1}d\rho+1 \\
&\quad+\int_0^{r}\frac{1}{w(r+\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma+1\right)^{-1}d\rho+1 \\
&\leq (1+C_0)J(r). \qquad\qquad \text{(VII.4.7)}
\end{aligned}
$$

In addition, with any $k \geq 0$ let $r_k > 0$ satisfy

$$\int_0^{r_k} \frac{1}{w(\rho)} d\rho = 2^k - 1,$$

where $r_k$ is determined uniquely, since $\int_0^r 1/w(\rho)d\rho$ is a monotone increasing function of $r$. Then, we estimate

$$\begin{aligned} J(r_k) &= \sum_{j=1}^{k} \int_{r_{j-1}}^{r_j} \frac{1}{w(\rho)} \left( \int_0^{\rho} \frac{1}{w(\sigma)} d\sigma + 1 \right)^{-1} d\rho \\ &\geq \sum_{j=1}^{k} \left( \int_0^{r_j} \frac{1}{w(\sigma)} d\sigma + 1 \right)^{-1} \left( \int_0^{r_j} \frac{1}{w(\rho)} d\rho - \int_0^{r_{j-1}} \frac{1}{w(\rho)} d\rho \right) \\ &= \sum_{j=1}^{k} 2^{-j}(2^j - 2^{j-1}) = \frac{k}{2}. \end{aligned} \tag{VII.4.8}$$

This shows that $J$ is unbounded. By (VII.4.4), (VII.4.5), and (VII.4.7), for any $x \in \mathbb{R}^n$ with $|x| \geq 4$ and $0 < \delta < 1/p$

$$\begin{aligned} &\int_{1 \leq |y| \leq |x|/2} w_0(|x+y|) f(x+y) w_2(|y|) h(y) dy \\ &= \int_{1 \leq |y| \leq |x|/2} \left( \int_0^{|x+y|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a} J(|x+y|)^{-1/p-\delta} \\ &\qquad \cdot (1+|y|)^{-(n-1)} \frac{1}{w(|y|)} \left( \int_0^{|y|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1} J(|y|)^{-1/p'-\delta} dy \\ &\geq C \left( \int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a} J(2|x|)^{-1/p-\delta} \\ &\qquad \cdot \int_1^{|x|/2} \frac{1}{w(r)} \left( \int_0^{r} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1} J(r)^{-1/p'-\delta} dr \\ &\geq C(1+C_0)^{-1/p-\delta} \frac{1}{1/p-\delta} \left( \int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a} J(|x|)^{-1/p-\delta} \\ &\quad \cdot \left( J(|x|)^{1/p-\delta} - J(1)^{1/p-\delta} \right) \\ &\geq C(1+C_0)^{-1/p-\delta} \frac{1}{1/p-\delta} \left( 1 - J(2)^{\delta-1/p} \right) \left( \int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a} \\ &\quad \cdot J(|x|)^{-2\delta} \end{aligned} \tag{VII.4.9}$$

with some positive constant $C$. Then, by (VII.4.9) and (VII.4.8), for $0 < \delta \leq 1/(3p)$, we estimate

$$
\begin{aligned}
&\iint w_0(|x+y|)w_1(|x|)w_2(|y|)f(x+y)g(x)h(y)\ dy\,dx \\
&\geq \int_{|x|\geq 4}\int_{1\leq|y|\leq|x|/2} w_0(|x+y|)w_1(|x|)w_2(|y|)f(x+y)g(x)h(y)\ dy\,dx \\
&\geq C(1+C_0)^{-1/p-\delta}\frac{1}{1/p-\delta}\Big(1-J(2)^{\delta-1/p}\Big) \\
&\quad\cdot\int_4^\infty \frac{1}{w(r)}\left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{-1} J(|x|)^{-1/p'-3\delta}dr \\
&\geq C(1+C_0)^{-1/p-\delta}\frac{1}{1/p-\delta}\Big(1-J(2)^{\delta-1/p}\Big)\Big(\lim_{r\to\infty}\log J(r)-\log J(4)\Big) \\
&=\infty,
\end{aligned}
$$

as required.

Q.E.D.

**Theorem 6.** *Let $w \in \mathscr{F}$ and let $w_0$, $w_1$, and $w_2$ be as in Theorem 2 with $a$, $b$, $c \in \mathbb{R}$.*
*(1) In the case I, the trilinear estimate in Theorem 2 holds for any $a$, $b$, $c \in \mathbb{R}$*

*(2) In the case II, let $a$, $b$, $c$ satisfy either (iii) or (iv) or (v) in Theorem 5, then the trilinear estimate in Theorem 2 fails for some $f \in L^\infty(\mathbb{R}^n)$, $g$, $h \in L^1(\mathbb{R}^n)$.*

*(3) In the case II, let $a = b = c = 0$. Then, the trilinear estimates holds.*

***proof***. The proofs of (1) and (2) are the same as in the proof of Theorem 5, while (3) follows from the Hölder and Young inequalities as below:

$$
\begin{aligned}
&\iint w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\ dx\,dy \\
&\leq \iint |f(x+y)g(x)h(y)|\ dx\,dy \\
&= \iint |f(x)g(x-y)h(y)|\ dy\,dx \\
&\leq \|f\|_{L^\infty}\|g*h\|_{L^1} \\
&\leq \|f\|_{L^\infty}\ \|g\|_{L^1}\ \|h\|_{L^1}.
\end{aligned}
$$

Q.E.D.

## References


[1] H. Ando, T. Horiuchi, and E. Nakai, "Construction of slowly increasing functions," Sci. Math. Japonicae Online, **e-2012**(2012), 207-221.

[2] H. Bahouri, J.-Y. Chemin, and R. Danchin, "Fourier Analysis and Nonlinear Partial Differential Equations," Springer, 2011.

[3] F.M. Christ and M.I. Weinstein, "Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation," J. Funct. Anal., **100**(1991), 87–109.

[4] P. D'Ancona, D. Foschi, and S. Selberg, "Atlas of products for wave-Sobolev space on $\mathbb{R}^{1+3}$," Trans. Amer. Math. Soc., **364**(2012), 31–63.

[5] J. Fan and T. Ozawa, "Regularity criterion for weak solutions to the Navier-Stokes equations in terms of the gradient of the pressure," J. Inequal. Appl., **2008**, Art. ID 412678, 6pp.

[6] A.E. Gatto, "Product rule and chain rule estimates for fractional derivatives on spaces that satisfy the doubling condition," J. Funct. Anal., **188**(2002), 27–37.

[7] J. Ginibre, T. Ozawa, and G. Velo, "On the existence of the wave operators for a class of nonlinear Schrödinger equations," Ann. Inst. H. Pincaré Pys. Théor., **60**(1994), 211–239.

[8] T. Kato, "On nonlinear Schrödinger equations, II. $H^s$ solutions and unconditional well-posedness," J. Anal. Math., **67**(1995), 281–306.

[9] T. Kato and G.Ponce, "Commutator estimates and the Euler and Navier-Stokes equations," Comm. Pure Appl. Math., **41**(1988), 891–907.

[10] S. Klainerman and S. Selberg, "Bilinear estimates and applications to nonlinear wave equations," Comm. Contemp. Math., **4**(2002), 223–295.

[11] H. Kozono and Y. Taniuchi, "Bilinear estimates in BMO and the Navier-Stokes equations," Math. Z., **235**(2000), 173–194.

[12] F. Linares and G. Ponce, "Introduction to Nonlinear Dispersive Equations," Springer, 2009.

[13] S. Machihara, K. Nakanishi, and K. Tsugawa, "Well-posedness for nonlinear Dirac equations in one dimension," Kyoto J. Math., **50**(2010), 403–451.

[14] T. Runst and W. Sickel, "Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations," Walter de Gruyter, 1996.

[15] J.-C. Saut and R. Temam, "Remarks on the Korteweg-de-Vries equation," Israel J. Math., **2014**(1976), 78–87.

[16] W. Sickel and H. Triebel, "Hölder inequalities and sharp embeddings in function spaces of $B^s_{pq}$ and $F^s_{pq}$ type," Z. Anal. Anwendungen, **14**(1995), 105–140.

[17] T. Tao, "Multilinear weighted convolutions of $L^2$ functions, and applications to nonlinear dispersive equations," Amer. J. Math., **123**(2001), 839–908.

[18] T. Tao, "Nonlinear Dispersive Equations, Local and Global Analysis," Amer. Math. Soc., 2006.

# List of Papers

[1] K. Fujiwara and T. Ozawa, “Exact remainder formula for the Young inequality and applications,” Int. Journal of Math. Analysis, **7**(2013), 2723 – 2735.

[2] K. Fujiwara, S. Machihara, and T. Ozawa, “Well-posedness for the Cauchy problem of a system of semirelativistic equations,” revised version submitted.

[3] K. Fujiwara, S. Machihara, and T. Ozawa, “Global well-posedness for the Cauchy problem of a system of semirelativistic equations,” to be submitted.

[4] K. Fujiwara and T. Ozawa, “Identities for the difference between the arithmetic and geometric means,” submitted.

[5] K. Fujiwara and T. Ozawa, “Stability of the Young and Hölder Inequalities,” submitted.

[6] K. Fujiwara and T. Ozawa, “Weighted $L^p$-boundedness of convolution type integral operators associated with bilinear estimates in the Sobolev spaces,” submitted.