Mathematical foundations of semirelativistic nonlinear fields

半相対論的非線形場の数学的基礎

October 2016

Research on Mathematical Physics
Department of Pure and Applied Physics
Graduate School of Advanced Science and Engineering
Waseda University

Kazumasa FUJIWARA
藤原 和将

Contents

Chapter 1

Introduction

1.1 Introduction

In this thesis, we study the Cauchy problem of the semirelativistic equation

$$\begin{cases} i\partial_t u \pm (m^2 - \Delta)^{1/2}\ u = F(u), & t \in [0, T),\ x \in \mathbb{R}, \\ u(0) = u_0, & x \in \mathbb{R}, \end{cases} \tag{SR}$$

where u is a complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}$, $F : \mathbb{C} \to \mathbb{C}$ denotes nonlinearity, $\partial_t = \partial/\partial t$, $m \in \mathbb{R}$, $\Delta = \partial_x^2 = (\partial/\partial x)^2$ is the Laplacian in $\mathbb{R}$. Moreover, the operator $(m^2 - \Delta)^{1/2}$ is defined as a Fourier multiplier with symbol $(m^2 + \xi^2)^{1/2}$. Namely, we define $(m^2 - \Delta)^{1/2} u = \mathfrak{F}^{-1}(m^2 + \xi^2)^{1/2}\mathfrak{F}u(\xi)$, where ξ is a real variable, $\mathfrak{F}$ is the Fourier transform defined by

$$\mathfrak{F}u(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x) \exp(-ix\xi) dx,$$

and $\mathfrak{F}^{-1}$ is the inverse Fourier transform defined by

$$\mathfrak{F}^{-1}u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(\xi) \exp(ix\xi) dx.$$

For simplicity, we describe $\mathfrak{F}u$ as $\hat{u}$.

The aim of this thesis is to study the solvability and well-posedness of the Cauchy problem (SR) with power type nonlinearity. Specifically, we study the relationship between the smoothness of initial data u_0 and the solvability and well-posedness of (SR) with some power type nonlinearities. Here, we say $u : [0, T) \to \mathscr{S}^*(\mathbb{R})$ is a time-local solution to the Cauchy problem (SR) for the initial data u_0 if u satisfies the following weak equation corresponding to (SR):

$$\int_0^T \langle u(t) \mid i\partial_t \phi + (m^2 - \Delta)^{1/2}\phi(t)\rangle_{\mathscr{S}(\mathbb{R})} dt = \langle u_0 \mid \phi(0)\rangle_{\mathscr{S}(\mathbb{R})} + \int_0^T \langle F(u(t)) \mid \phi(t)\rangle_{\mathscr{S}(\mathbb{R})} dt$$

for any $\phi \in \mathscr{S}(\mathbb{R}^2)$, where $\mathscr{S}(\mathbb{R}^n)$ is the set of all smooth rapidly decreasing functions, $\mathscr{S}^*(\mathbb{R}^n)$ is the dual of $\mathscr{S}(\mathbb{R}^n)$, and $\langle \cdot \mid \cdot \rangle_{\mathscr{S}(\mathbb{R})}$ is the dual product of $\mathscr{S}(\mathbb{R})$. We also say $u : \mathbb{R} \to \mathscr{S}'(\mathbb{R})$ is a time-global solution to the Cauchy problem (SR) for the initial data u_0 if u is a time-local solution for any $T \in \mathbb{R}$. Let $X \subset \mathscr{S}^*(\mathbb{R})$ be a Banach space. We say the Cauchy problem (SR) is time-locally well-posed with respect to X, if any initial data $u_0 \in X$, we have some $T \in \mathbb{R}$ which depend only on $\|u_0\|_X$ such that we have a unique solution $u \in C([0, T); X)$ to (SR) and the solution map from X to $C([0, T); X)$ is continuous. We also say (SR) is ill-posed if (SR) is not time-locally well-posed and (SR) is time-globally well-posed if (SR) is time-locally well-posed for any $T \in \mathbb{R}$. In this thesis, we consider nonlinearity of the following forms:

$$F(z) = \lambda |z|^p, \quad \lambda |z|^{p-1} z, \quad \lambda z^p, \quad \lambda \overline{z}^p,$$

where $p \geq 1$, λ is a non-zero complex number and $\overline{z}$ is a complex conjugate of z. It is widely known that for general differential equations, the condition of initial data for the solvability and well-posedness depends on form of nonlinearity, λ, and p. So the aim of this thesis is to study the sharp condition of initial data for the solvability and well-posedness of (SR) for each form, λ and p.

1.2 Physical Background

To motivate our problem, we revisit four fundamental equations with regard to quantum and relativistic quantum mechanics: the Schrödinger, Klein-Gordon, Dirac, and semirelativistic equations. The Schrödinger equation is the first model to describe quantum particles. The origin of the Schrödinger equation is the following non-relativistic energy equation of a free particle,

$$E = \frac{p^2}{2m}, \tag{1.2.1}$$

where E is energy, p is momentum, and m is mass of free particle. The free Schrödinger equation is derived by quantizing (1.2.1) as follows:

$$E = i\hbar \frac{\partial}{\partial t}, \quad p = -i\hbar \nabla,$$

On the other hand, in special relativity, the relativistic energy equation of a free particle is described by

$$E = \sqrt{m^2 c^4 + |p|^2 c^2}, \tag{1.2.2}$$

where c is the speed of light. By quantizing (1.2.2), the following free semirelativistic equation is obtained:

$$i\hbar \frac{\partial}{\partial t} \psi = (m^2 c^4 - c^2 \hbar^2 \Delta)^{1/2} \psi.$$

Although, the semirelativistic equation is naturally derived by simple quantizing of the relativistic energy of a free particle, the semirelativistic equation had not been considered as a fundamental equation of a free relativistic particle, since the operator $(1 - \Delta)^{1/2}$ is non-local. To avoid the non-local operator, free Klein-Gordon equation is introduced by quantizing the squared energy:

$$E^2 = m^2 c^4 + |p|^2 c^2.$$

However, since the Klein-Gordon equation is second order in time, the free Klein-Gordon equation does not admit a positive definite energy. To obtain the definite relativistic density and energy, the free Dirac equation has been considered as a modification of the free Klein-Gordon equation. By this modification, the energy of the free Dirac equation is definite but negative definite for some initial data. To justify this non-positive definiteness, many physical ideas have been considered. However, we may now have enough mathematical knowledge to consider semirelativistic equations directly without any modification of non-local operator. So in this thesis, we consider some fundamental properties of semirelativistic equation.

The semirelativistic equations have been used also in other physical model. For example, with Hartree type nonlinearity, the semirelativistic equation is used to describe boson stars. For the details of the model boson stars, we refer the reader to [22, 29, 66, 67]. Moreover, the mass-less semirelativistic equation is obtained by Laskin by the generalized Feynman path integral. In this case, the mass-less semirelativistic equation is called as fractional Schrödinger equation. For the details of the relationship between fractional Schrödinger equation and generalized path integral, we refer the reader to [46, 63, 64, 65]. Even in non-quantum physics, semirelativistic equation has been used to describe physical phenomenon. For instance, the mass-less semirelativistic equation with power type nonlinearity is used as a model of wave turbulence. In this case, semirelativistic equation is called as half wave equation. For the details of the model of wave turbulence, we refer the reader to [18, 62, 74].

In this thesis, we consider the semirelativistic equation with power type nonlinearity, since power type nonlinearity may be considered as one of the most fundamental nonlinearity and used as an approximation of general nonlinearity. So, this thesis is devoted to understand foundation of the nonlinear semirelativistic equation.

1.3 Preliminaries

In this section, we prepare some basic knowledge to state our main results and discuss further.

1.3.1 Banach Space

Let X and Y be Banach spaces over $\mathbb{C}$. We define the norm of $X \cap Y$ as

$$\|u\|_{X\cap Y} = \|u\|_X + \|u\|_Y.$$

We denote the dual of X as X^*. We call X as a reflexive Banach space, if X is identified with X^{**}.

The following lemmas are essential statements. For the details, for instance, we refer the reader to [16].

Lemma 1.3.1. *Moreover, Let $k \in \mathbb{N}$ and $(X_j)_{j=1}^k$ is a sequence of Banach spaces. Let T is a k-linear operator from $\prod_{j=1}^k X_j$ into Y. Then there exists $C > 0$ such that for any $x \in \prod_{j=1}^k X_j$,*

$$\|T(x)\|_Y \le C \prod_{j=1}^{k} \|x_j\|_{X_j}.$$

Lemma 1.3.2. *Let X be a reflexive Banach space and $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of X. Then there exists a weakly convergent subsequence of $(x_n)_{n\in\mathbb{N}}$. Moreover, let x is a weak limit of $(x_n)_{n\in\mathbb{N}}$, then $\|x\|_X \le \liminf_n \|x_n\|_X$.*

Here, we also prepare fundamental argument of quotient spaces. Let X be a Banach space and $M \subset X$ be a subspace. Then for any $x \in X$, we denote an equivalent class of x as $[x]$ and define $[x]$ as

$$[x] = \{y \in X \ \mid \ x - y \in M\}.$$

We also denote the set of all equivalent classes in X with regard to M as X/M and define the norm of X/M as

$$\|[x]\|_{X/M} = \inf_{y\in[x]} \|y\|_X = \inf_{z\in M} \|x + z\|_X.$$

The following lemma is also basic and we use it in Chapter 2.

Lemma 1.3.3. *Let X be a Banach space and M be a closed subspace. Then X/M is also a Banach space.*

We say a normed space X is embedded into another normed space Y, if $X \subset Y$ and there exists $C > 0$ such that

$$\|x\|_Y \leq C\|x\|_X$$

for any $x \in X$ and denote $X \hookrightarrow Y$.

Next, we consider the following X-valued equation

$$x = L(x_0) + N_k(x, \cdots, x) \tag{1.3.1}$$

with $x_0 \in X_0$, where X_0 is a Banach space, $L : X_0 \to X$ is a linear map and $N : X^k \to X$ is a k-linear map. We say (1.3.1) is quantitatively well-posed in X_0 and X if L and N satisfy

$$\|L(x_0)\|_X \leq C\|x_0\|_{X_0}, \quad \|N((x_j)_{j=1}^k)\|_X \leq C \prod_{j=1}^k \|x_j\|_X.$$

The following contraction argument is basic to construct solutions for (1.3.1).

Lemma 1.3.4 ([3]). *Let (1.3.1) be quantitatively well-posed in X_0 and X. Then there exists C_0 and $\varepsilon > 0$ such that for any x with $\|x\|_X < \varepsilon$, we have a unique solution x satisfying $\|x\|_X < C_0\varepsilon$ and*

$$x = \sum_{j=1}^{\infty} A_j(x_0), \tag{1.3.2}$$

where

$$\begin{aligned} A_1(x_0) &= L(x_0), \\ A_j(x_0) &= \sum_{j_1 \cdots j_k \geq 1, \sum_{l=1}^k j_l = j} N_k((A_{j_l}(x_0))_{l=1}^k). \end{aligned}$$

If $A_j(x_0) \notin X$ for some j, it doesn't imply that there exists no solutions for (1.3.1) but it is impossible to construct a solution by (1.3.2) successively. If A_j is not continuous map form X_0 to X for some j, the solution map for (1.3.1) is called not C^j, since

$$\frac{d^j}{d\rho^j} \sum_{j'=1}^{\infty} A_{j'}(\rho x_0)\Big|_{\rho=0} = \frac{d^j}{d\rho^j} \sum_{j'=1}^{\infty} \rho^{j'} A_{j'}(x_0)\Big|_{\rho=0} = j! A_j(x_0).$$

Although the discontinuity of some A_j seems not sufficient to show the discontinuity of the solution map, under an appropriate condition, we can show the discontinuity of the solution map from the discontinuity of some A_j.

Lemma 1.3.5 ([3]). *Let (1.3.1) be quantitatively well-posed in X_0 and X. Let $X_0 \hookrightarrow Y_0$ and $X \hookrightarrow Y$. Let the solution map of (1.3.1) $u_0 \to u$ be continuous from $B_{X_0}(r) = \{u_0 \in X_0 \mid \|u_0\|_{X_0} < r_0\}$ with $\|\cdot\|_{Y_0}$ into $\{u \in X \mid \|u\|_X < r\}$ with $\|\cdot\|_Y$ for some $r_0, r > 0$. Then each A_j is a continuous map form $B_{X_0}(r) = \{u_0 \in X_0 \mid \|u_0\|_{X_0} < r_0\}$ with $\|\cdot\|_{Y_0}$ into X with $\|\cdot\|_Y$.*

1.3.2 Lebesgue Space

Here, we collect some basic statements of the Lebesgue space. Let $n \in \mathbb{N}$ and let X be a Banach space. Let $1 \leq p \leq \infty$. For a measurable X valued function f on $D \subset \mathbb{R}^n$, we denote the L^p norm of f as $\|f\|_{L^p(D;X)}$ and define $\|f\|_{L^p(D;X)}$ as

$$\|f\|_{L^p(D;X)} = \begin{cases} \left(\int_D \|f(x)\|_X^p dx \right)^{1/p}, & \text{if} \quad 0 < p < \infty, \\ \inf\{C \in \mathbb{R} \mid \|f\|_X \leq C, \text{ a.e. on } D\}, & \text{if} \quad p = \infty \end{cases}$$

and we denote $L^p(D;X)$ is all X valued measurable functions whose L^p norm is finite. We also define $L^1_{\mathrm{loc}}(\mathbb{R}^n;X)$ as a set of all measurable functions $f:\mathbb{R}^n\to X$ such that $f\in L^1(D;X)$ for any compact set D. It is widely known that if X is a Banach space, then $L^p(D;X)$ is a Banach space for any $1\le p\le\infty$. It is also basic fact that if X is a reflexive Banach space, then $L^p(D;X)$ is a reflexive Banach space for any $1<p<\infty$ and the dual space of $L^p(D;X)$ is identified with $L^{p'}(D;X^*)$, where

$$p'=\begin{cases}\infty & \text{if} \quad p=1,\\ \frac{p}{p-1} & \text{if} \quad 1<p<\infty,\\ 1 & \text{if} \quad p=\infty.\end{cases}$$

For simplicity, we abbreviate $L^p(D;\mathbb{C})$ as $L^p(D)$. Moreover, for complex valued functions f and g on $\mathbb{R}^n$, we denote the convolution of f and g as $f*g$ and define $f*g$ as

$$f*g(x)=\int_{\mathbb{R}^n}f(x-y)g(y)\,dy.$$

Here, we collect 4 more basic facts. For the details of them, for instance, we refer the reader to [16, 33].

Lemma 1.3.6. *Let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If for any $\phi\in C_c^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n}f(x)\phi(x)dx=0,$$

then $f=0$ a.e., where $C_c^\infty(\mathbb{R}^n)$ is the set of all compact supported smooth functions on $\mathbb{R}^n$.

Lemma 1.3.7 (the Hölder inequailty)**.** *Let $D\subset\mathbb{R}^n$. Let $p,q,r\in[1,\infty]$ satisfy $\dfrac{1}{p}=\dfrac{1}{q}+\dfrac{1}{r}$. Then, for measurable functions f,g on D,*

$$\|fg\|_{L^p(D)}\le\|f\|_{L^q(D)}\|g\|_{L^r(D)},$$

where the equality holds if $f=g$.

Lemma 1.3.8 (the Young inequality)**.** *Let $p,q,r\in[1,\infty]$ satisfy $1+\dfrac{1}{p}=\dfrac{1}{q}+\dfrac{1}{r}$. Then , for measurable functions f,g on $\mathbb{R}^n$, the following estimate holds:*

$$\|f*g\|_{L^p(\mathbb{R}^n)}\le\|f\|_{L^q(\mathbb{R}^n)}\|g\|_{L^r(\mathbb{R}^n)}.$$

Lemma 1.3.9 (the Hausdorf-Young ineqality)**.** *Let $p\in[2,\infty]$. Then, for measurable functions f,g on $\mathbb{R}^n$, the following estimate holds:*

$$\|\mathfrak{F}f\|_{L^p(\mathbb{R}^n)}\le\|f\|_{L^{p'}(\mathbb{R}^n)}.$$

Moreover, the Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$.

1.3.3 Sobolev Space

Here, we collect some basic statements of the Sobolev space. For $s\in\mathbb{R}$ and $1\le p\le\infty$, we define the inhomogeneous Sobolev space of order s based on $L^p(\mathbb{R}^n)$ as

$$\{f\in\mathscr{S}^*(\mathbb{R}^n)\ \mid\ (1-\Delta)^{s/2}f\in L^p(\mathbb{R}^n)\},$$

and denote it as $H^s_p(\mathbb{R}^n)$, where $(1-\Delta)^{s/2}$ is a Fourier multiplier with symbol $(1+|\xi|^2)^{s/2}$, namely,

$$(1-\Delta)^{s/2}f=\mathfrak{F}^{-1}(1+|\xi|^2)^{s/2}\mathfrak{F}f.$$

For simplicity, we denote $(1+|\xi|^2)^{1/2}$ as $\langle\xi\rangle$. We also define the norm of $H_p^s(\mathbb{R}^n)$ as

$$\|f\|_{H_p^s(\mathbb{R}^n)} = \|(1-\Delta)^{s/2} f\|_{L^p(\mathbb{R}^n)}.$$

Similarly, we also define the homogeneous Sobolev space of order s based on $L^p(\mathbb{R}^n)$ as

$$\{f \in \mathscr{S}^*(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n) \ \mid\ (-\Delta)^{s/2} f \in L^2(\mathbb{R}^n)\}$$

and denote it as $\dot{H}_p^s(\mathbb{R}^n)$, where $\mathscr{P}(\mathbb{R}^n)$ is the set of all polynomials on $\mathbb{R}^n$ and $(-\Delta)^{s/2}$ is a Fourier multiplier with symbol $|\xi|^s$. We define the norm of $\dot{H}_p^s(\mathbb{R}^n)$ as

$$\|f\|_{\dot{H}_p^s(\mathbb{R}^n)} = \|(-\Delta)^{s/2} f\|_{L^p(\mathbb{R}^n)}.$$

For simplicity, we denote $H_2^s(\mathbb{R}^n)$ as $H^s(\mathbb{R}^n)$ and $\dot{H}_2^s(\mathbb{R}^n)$ as $\dot{H}^s(\mathbb{R}^n)$.

The following estimates are basic in this thesis:

Lemma 1.3.10 ([6, 88] the Sobolev embedding)**.** *If $s_1 - n/p_1 = s_2 - n/p_2$, $1 < p_1 < p_2 < \infty$, and $s_1 > s_2$, $\dot{H}_{p_1}^{s_1}(\mathbb{R}^n) \hookrightarrow \dot{H}_{p_2}^{s_2}(\mathbb{R}^n)$ and $H_{p_1}^{s_1}(\mathbb{R}^n) \hookrightarrow H_{p_2}^{s_2}(\mathbb{R}^n)$. Moreover, if $s - n/p > 0$, $\dot{H}_p^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ and $H_p^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$*

Lemma 1.3.11. *For $f, g \in \mathscr{S}(\mathbb{R}^n)$ and $a, b, c \in \mathbb{R}$, the estimates*

$$\|fg\|_{H^{-a}(\mathbb{R}^n)} \le C\|f\|_{H^b(\mathbb{R}^n)}\|g\|_{H^c(\mathbb{R}^n)} \tag{1.3.3}$$

holds if and only if

$$a+b+c > \frac{n}{2}, \quad a+b \ge 0, \quad b+c \ge 0, \quad \text{and} \quad c+a \ge 0,$$

or

$$a+b+c \ge \frac{n}{2}, \quad a+b > 0, \quad b+c > 0, \quad \text{and} \quad c+a > 0.$$

Especially, when $-a = b = c = s$, (1.3.3) holds with $s > n/2$. A simple proof for the sharp sufficient condition (1.3.3) is shown in chapter A.2 and an improved estimate with $-a = b = c$ is argued in Appendix A.3.

The following lemma is also useful to extend the time-local solution.

Lemma 1.3.12 ([15, 17, 81])**.** *Let $s > n/2$. There exists $C = C(n, s)$ such that for $f \in H^s(\mathbb{R}^n)$,*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le C\|f\|_{H^{n/2}(\mathbb{R}^n)}\sqrt{\log(1+\|f\|_{H^s(\mathbb{R}^n)})} + 1.$$

1.3.4 Besov Space

Here, we collect some basic statements of the Besov space. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ satisfy $\hat{\phi} \ge 0$ and supp $\hat{\phi} \subset \{\xi \in \mathbb{R}^n | 1/2 < |\xi| < 2\}$ and

$$\sum_{j=-\infty}^{\infty} \hat{\phi}(2^{-j}\xi) = 1$$

if $\xi \ne 0$. We denote ϕ_j as $2^{jn}\phi(2^j\cdot)$. Then, for $s \in \mathbb{R}$, $1 \le p, q \le \infty$, we define the inhomogeneous Besov space of order s based on $L^p(\mathbb{R}^n)$ as

$$\{f \in \mathscr{S}^*(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n) \ \mid\ (2^{sj}\|\phi_j * f\|_{L^p(\mathbb{R}^n)})_{j\in\mathbb{Z}} \in l^q\},$$

and denote it as $\dot{B}_{p,q}^s(\mathbb{R}^n)$. We also define the norm of $\dot{B}_{p,q}^s(\mathbb{R}^n)$ as

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} = \|2^{sj}\|\phi_j * f\|_{L^p(\mathbb{R}^n)}\|_{l_j^q}.$$

Lemma 1.3.13 ([6]). *For any* $s \in \mathbb{R}$ *and* $1 \le p \le \infty$, $\dot{B}^s_{p,q_0} \hookrightarrow \dot{B}^s_{p,q_1}$ *if* $1 \le q_0 < q_1 \le \infty$. *Moreover, for any* $s \in \mathbb{R}$, $\dot{H}^s(\mathbb{R}^n) \sim \dot{B}^s_{2,2}(\mathbb{R}^n)$.

Lemma 1.3.14 ([35]). *Let* $F \in C^1(\mathbb{C}, \mathbb{C})$ *satisfy* $F(0) = F_z(0) = F_{\overline{z}}(0) = 0$ *and assume that for* $p \ge 1$,

$$\begin{aligned} &\max(|F_z(z_1) - F_z(z_2)|, |F_{\overline{z}}(z_1) - F_{\overline{z}}(z_2)|) \\ &\le \begin{cases} C|z_1 - z_2| \max(|z_1|, |z_2|)^{p-2} & \text{if} \quad p \ge 2, \\ C|z_1 - z_2|^{p-1} & \text{if} \quad 1 < p < 2 \end{cases} \end{aligned} \tag{1.3.4}$$

for all $z_1, z_2 \in \mathbb{C}$, *where* $F_z = \frac{1}{2}(\frac{\partial}{\partial x}F - i\frac{\partial}{\partial y}F)$ *and* $F_{\overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x}F + i\frac{\partial}{\partial y}F)$ *with* $x = \operatorname{Re} z$ *and* $y = \operatorname{Im} z$. *Let* $0 \le s < \min\{2, p\}$ *and* $1 \le l, r, q \le \infty$ *with* $(p-1)/q = 1/l - 1/r$. *Then*

$$\|F(f)\|_{\dot{B}^s_{l,2}(\mathbb{R})} \le C\|f\|_{\dot{B}^s_{r,2}(\mathbb{R})}\|f\|^{p-1}_{L^q(\mathbb{R})}.$$

We remark that $|z|^{p-1}z$ and $|z|^p$ satisfy (1.3.4) since

$$\begin{aligned} \frac{\partial}{\partial x}|z|^{p-1}z &= |z|^{p-1} + (p-1)x|z|^{p-3}z, \\ \frac{\partial}{\partial y}|z|^{p-1}z &= i|z|^{p-1} + (p-1)y|z|^{p-3}z, \\ \frac{\partial}{\partial x}|z|^p &= px|z|^{p-2}, \\ \frac{\partial}{\partial y}|z|^p &= py|z|^{p-2}, \end{aligned}$$

and with $a \in \mathbb{R}$

$$\begin{aligned} ||z_1|^a - |z_2|^a| &= \left| a\int_0^1 ((1-\theta)|z_1| + \theta|z_2|)^{a-1} d\theta(|z_1| - |z_2|) \right| \\ &\le (|z_1| + |z_2|)^{a-1}|z_1 - z_2|. \end{aligned}$$

1.4 Earlier Works

Here, we introduce some earlier works for the well-posedness of the Cauchy problem for semirelativistic equation with power type nonlinearity in the case of $\mathbb{R}$. By using the Duhamel's formula, (SR) is rewritten into the following integral equation:

$$u(t) = U(\pm t)u_0 - i\int_0^t U(\pm(t-t'))F(u(t'))dt', \tag{ISR}$$

where $U(t)$ is a semirelativistic propagator defined by $U(t) = \exp(it(m^2 - \Delta)^{1/2})$. If $F(z) = \lambda|z|^{p-1}z$ or $\lambda|z|^p$ with $p \ge 1$ and $1/2 < s < \min(2, p)$, then for any λ, (ISR) is time-locally well-posed. Moreover, if $F(z) = \lambda z^a \overline{z}^b$ with non-negative integers a, b and $s > 1/2$, then for any λ, (ISR) is time-locally well-posed. These time-local well-posedness are obtained by standard contraction argument based on $H^s(\mathbb{R})$. In particular, we can show that the solution map of (ISR) is a contraction map on $H^s(\mathbb{R})$ by the unitarity of $U(t)$ in $H^s(\mathbb{R})$ and Lemmas 1.3.10, 1.3.11, and 1.3.14. Moreover, in [15], Borgna and Rial showed that if $F(z) = |z|^2 z$ and $s > 1/2$, then (ISR) is time-globally well-posed. In [62], Krieger, Lenzmann, and Raphaël showed that if $F(z) = |z|^2 z$ and $s \ge 1/2$, then (ISR) is time-globally well-posed. In [15, 62], they extend time-local $H^s(\mathbb{R})$ valued solutions with $s > 1/2$ by conserved energy and Lemma

1.3.12. Moreover, Krieger, Lenzmann, and Raphaël obtained $H^{1/2}(\mathbb{R})$ valued time-global solutions by limits of sequences of smooth approximating solutions. However, we remark that contraction argument based on Lemmas 1.3.10, 1.3.11, and 1.3.14 is not applicable in the $H^s(\mathbb{R}^n)$ setting with $s \leq n/2$, where $n \in \mathbb{N}$. It is because that although the uniform control of solutions is necessary for the construction, it is impossible to control solutions uniformly only by $H^s(\mathbb{R}^n)$ norm of solutions with $s \leq n/2$.

On the other hand, for $F(z) = \lambda|z|^p$ or $\lambda|z|^{p-1}z$ or λz^p or $\lambda\overline{z}^p$, (SR) is invariant under the following scaling transformation:

$$u_\rho(t,x) = \rho^{\frac{1}{p-1}} u(\rho t, \rho x)$$

with $\rho > 0$. Then $p_{n,s}^{(SR)}$ is called a scaling critical exponent corresponding to $\dot{H}^s(\mathbb{R}^n)$ if

$$\|u_\rho(0)\|_{\dot{H}^s(\mathbb{R}^n)} = \rho^{\frac{1}{p-1}+s-\frac{n}{2}} \|u_0\|_{\dot{H}^s(\mathbb{R}^n)} = \|u_0\|_{\dot{H}^s(\mathbb{R}^n)}$$

for any $\rho > 0$ with $p = p_{n,s}^{(SR)}$. In this case,

$$p_{n,s}^{(SR)} = 1 + \frac{2}{n-2s}$$

and the regularity $s_{n,p}^{(SR)}$ corresponding to $p_{n,s}^{(SR)}$ is given by

$$s_{n,p}^{(SR)} = \frac{n}{2} - \frac{1}{p-1}. \tag{1.4.1}$$

Based on this scaling criticality, we classify Cauchy problems into three. A Cauchy problem is called H^s supercritical if $p > p_{n,s}^{(SR)}$. We call a Cauchy problem H^s critical if $p = p_{n,s}^{(SR)}$. A Cauchy problem is also called H^s subcritical if $p < p_{n,s}^{(SR)}$. From the view point of the regularity, we call our problem supercritical if $s < s_{n,p}^{(SR)}$, critical if $s = s_{n,p}^{(SR)}$, subcritical if $s > s_{n,p}^{(SR)}$. On the analogy of the Schrödinger and Klein-Gordon equations, in subcritical and critical cases, (SR) is expected time-locally well-posed. From the view point of scaling criticality, in [55], Inui showed that for $F(z) = \lambda|z|^p$, in H^s subcritical and critical cases, there exists no time-global solutions for some H^s initial data. He also shows that in H^s supercritical, there exists no time-local solutions for some H^s initial data. However, in some subcritical and critical case, the solvability, well-posedness, and ill-posedness of (SR) had not been shown.

1.5 Main Statement

The aim of this thesis is to obtain the lowest regularity s with which (SR) is solvable or well-posed in the frame work of $H^s(\mathbb{R})$ in one spacial dimension case. Since (SR) has the invariant scaling transformation, the scaling critical exponent seems to give the sharp criteria for the solvability and time-local well-posedness. However, at least in the case of $\mathbb{R}$, there is a gap between $s_{n,p}^{(SR)}$ and $1/2$ with which (SR) is proved to be time-globally well-posed in the prior works. To consider the solvability and well-posedness for $s_{n,p} \leq s \leq 1/2$, we need more sharp linear estimate for $U(t)$, a priori estimate of energy, and nonlinear estimate for $F(z)$.

At first, to obtain the solvability and well-posedness of (SR) for $s < 1/2$, we focus on nonlinear interaction. Here, in order to consider a sharp nonlinear estimate with simple nonlinearities, we put $F(z) = \lambda z^a \overline{z}^b$ with $\lambda \in \mathbb{C}\backslash\{0\}$ and $(a,b) = (2,0)$ or $(1,1)$ or $(0,2)$. Then, we have the following sharp criteria to construct time-local solutions by iteration scheme.

Theorem 1. (SR) *with* $\lambda \in \mathbb{C}\backslash\{0\}$ *and* $F(z) = \lambda\overline{z}^2$ *is time-locally well-posed if* $s \geq 0$. *If* $-1/2 < s < 0$, *then for initial data* u_0 *satisfying*

$$u_0 = \mathfrak{F}^{-1}[\chi_{[1,\infty)}\langle\cdot\rangle^{-1/2-s-\varepsilon}] \in H^s(\mathbb{R})\backslash L^2(\mathbb{R})$$

with $0 < \varepsilon < -2s/3$,

$$\int_0^t U(\pm(t-t'))\left(\overline{U(\pm t')u_0}\int_0^{t'} U(\pm((t''-t'))(U(\pm t')u_0)^2 dt''\right) dt' \notin H^s(\mathbb{R}). \tag{1.5.1}$$

Therefore, the solution map of (SR) *with* $F(z) = \lambda\overline{z}^2$ *is not* C^3. *If* $s < -1/2$, *then for a sequence of initial datum* $u_{0,k}$ *defined by*

$$u_{0,k} = n^{-s}\mathfrak{F}^{-1}[\chi_{[-1,1]}(\cdot - k) + \chi_{[-1,1]}(\cdot + k)],$$

there exists $C > 0$ *such that for any* k, $\|u_{0,k}\|_{H^s} \le C$ *and for some* $t > 0$,

$$\limsup_{k\to\infty}\left\|\int_0^t U(\pm(t-t'))\overline{U(\pm t')u_{0,k}}^2 dt'\right\|_{H^s(\mathbb{R})} = \infty. \tag{1.5.2}$$

Therefore, the solution map of (SR) *with* $F(z) = \lambda\overline{z}^2$ *is not* C^2.

In addition, if $-1/2 < s < 1/2$, *then for initial data* u_0 *satisfying*

$$u_0 = \mathfrak{F}^{-1}[\chi_{[0,\infty)}\langle\cdot\rangle^{-1/2-s-\varepsilon}] \in H^s(\mathbb{R})\backslash H^{1/2}(\mathbb{R})$$

with $0 < \varepsilon < 1/6 - s/3$,

$$\int_0^t U(\pm(t-t'))\big(\overline{U(\pm t')u_0}U(\pm t')u_0\big)dt' \notin H^s(\mathbb{R}), \tag{1.5.3}$$

$$\int_0^t U(\pm(t-t'))\big((U(\pm t')u_0)^2\big)dt' \notin H^s(\mathbb{R}). \tag{1.5.4}$$

Therefore, the solution map of (SR) *with* $F(z) = \lambda|z|^2$ *and* λz^2 *are not* C^2. *If* $s < -1/2$, *then for a sequence of initial datum* $u_{0,k}$ *defined by*

$$u_{0,k} = n^{-s}\mathfrak{F}^{-1}[\chi_{[-1,1]}(\cdot - k)]$$

and for some $t > 0$

$$\limsup_{k\to\infty}\left\|\int_0^t U(\pm(t-t'))\big(\overline{U(\pm t')u_{0,k}}U(\pm t')u_{0,k}\big)dt'\right\|_{H^s(\mathbb{R})} = \infty, \tag{1.5.5}$$

$$\limsup_{k\to\infty}\left\|\int_0^t U(\pm(t-t'))\big((U(\pm t')u_0)^2\big)dt'\right\|_{H^s(\mathbb{R})} = \infty. \tag{1.5.6}$$

Therefore, the solution map of (SR) *with* $F(z) = \lambda|z|^2$ *and* λz^2 *are not* C^2.

Remark 1.5.1. *We remark that* (1.5.1), (1.5.2), (1.5.3), (1.5.4), (1.5.5), *and* (1.5.6) *imply that the associated solution maps are not* C^3, C^2, C^2, C^2, C^2, *and* C^2 *respectively, since we can regard* (SR) *as* (1.3.1) *with* $L(u_0)(t) = U(\pm t)u_0$ *and*

$$N(u)(t) = \int_0^t U(\pm(t-t'))F(u(t'))dt'.$$

With $X_0 = H^s(\mathbb{R})$, $X = C([0,T];H^s(\mathbb{R}))$, *these discontinuity results are obtained.*

(SR) *with* $F(z) = \lambda\overline{z}^2$ *is quantitatively well-posed in* $L^2(\mathbb{R})$ *and* $C([0,T];L^2(\mathbb{R}))$ *for some* $T > 0$. (SR) *with* $F(z) = \lambda z^2$ *and* $\lambda|z|^2$ *are also quantitatively well-posed in* $H^s(\mathbb{R})$ *and* $C([0,T];H^s(\mathbb{R}))$ *for* $s > 1/2$ *and some* $T > 0$. *However, Lemma 1.3.5 is not applicable to these Cauchy problems to show their ill-posedness with the sequences of initial data above. It is because the sequences of initial data in Theorem 1 are not bounded in associated spaces of initial data, in which* (SR) *is quantitatively well-posed.*

The well-posedness in Theorem 1 follows from a sharp bilinear estimate of the Fourier restriction norm. On the other hand, (1.5.1), (1.5.2), (1.5.3), (1.5.4) (1.5.5), and (1.5.6) follow from direct calculations. Especially to calculate (1.5.1) and (1.5.3), we use a special cancellation of spacial frequency. This cancellation makes (SR) with quadratic nonlinearity not be solvable with iteration argument even in the scaling subcritical case, where $s_{1,2}^{(SR)} = -1/2$. Moreover, a similar phenomena of Theorem 1 occurs in the case of the Cauchy problem for semilinear systems. We discuss the case of semilinear systems in Appendix A.1. We remark that the smoothness of solution maps in the $H^{-1/2}(\mathbb{R})$ setting is still not shown.

At second, we revisit the construction of solutions by a priori energy estimate. Here, we put $F(z) = \lambda |z|^{p-1} z$ with $1 < p \leq 3$ and consider (SR) in the $H^{1/2}(\mathbb{R})$ setting. We remark that for any $n \in \mathbb{N}$, the charge and energy of (SR) with $F(z) = \lambda |z|^{p-1} z$ correspond to $L^2(\mathbb{R}^n)$ and $H^{1/2}(\mathbb{R}^n)$, respectively. We also remark that in the $H^s(\mathbb{R})$ setting with $s > 1/2$, the time-local well-posedness follows from the unitarity of $U(\pm t)$ and the Sobolev embedding $L^\infty \hookrightarrow H^s(\mathbb{R})$. On the other hand, in the $H^{1/2}(\mathbb{R})$ setting, it is impossible to show the time-local well-posedness by the unitarity of $U(\pm t)$ since $H^{1/2}(\mathbb{R})$ is not embedded into $L^\infty(\mathbb{R})$. However, by using the energy conservation, we obtain the following well-posedness result.

Theorem 2. (SR) *with* $F(z) = \lambda |z|^{p-1} z$ *is time-globally well-posed in the* $H^{1/2}(\mathbb{R})$ *setting if*

- $1 < p \leq 3$ *and* $\lambda \leq 0$,
- $1 < p < 3$ *and* $\lambda > 0$,
- $p = 3$, $\lambda > 0$, *and* $\|u\|_{L^2(\mathbb{R})} \ll 1$.

Remark 1.5.2. *The condition in Theorem 2 is is for controlling* $H^{1/2}(\mathbb{R})$ *norm of solutions uniformly in time by the conserved energy.*

Theorem 2 follows from a priori estimate of solutions. Kenig, Ponce, and Vega obtained the same result when $p = 3$ by using the compactness argument based on Lemma 1.3.2. On the other hand, Theorem 2 may be shown by more direct way based explicitly on completeness of $L^2(\mathbb{R})$. To simplify the construction of solutions, here, the Yosida type smoothness operator plays an important role. In particular, a sequence of approximation solutions connected with Yosida type smoothness operator is shown to be a Cauchy sequence in $L^2(\mathbb{R})$ and the limit of the sequence is a time-global solution.

We remark that if $F(z) = \lambda |z|^2 z$, then from a similar calculation to (1.5.1), the solution map is shown to be not C^3 in the $H^s(\mathbb{R})$ setting with $-1/2 < s < 1/2$. This means, in this case, $H^{1/2}(\mathbb{R})$ gives the sharp criteria so that for any $H^s(\mathbb{R})$ initial data, the associated time local solutions can be obtained by iteration scheme. We also remark that $s_{1,3}^{(SR)} = 0$.

At last, we consider the solvability of (SR) with $F(z) = \lambda |z|^p$. In Theorem 1, it is asserted that if $p = 2$ and $s \in (-\infty, -1/2) \cup (-1/2, 1/2)$, then it is impossible to construct solutions by a standard iteration scheme in the $H^s(\mathbb{R})$ setting. But Theorem 1 doesn't imply that there is no time-local solutions in this case. We remark that for any $p > 1$, $s_{1,p}^{(SR)} < 1/2$. In supercritical case: $s < s_{1,p}^{(SR)}$, Inui showed that there exist no weak time-local solutions for some $H^s(\mathbb{R})$ initial data in [55], but if $s_{1,p}^{(SR)} < s < 1/2$, then there expected to be time-local solutions to (SR). Here, we show that there exist no weak time-local solutions with some $H^{1/2}(\mathbb{R})$ initial data. To state our statement clearly, we define the weak time-local solutions for (SR) with $F = \lambda |z|^p$. For $T > 0$, we define function spaces A and A_T for $T > 0$ as follows:

$$A = C([0, \infty); H^2(\mathbb{R}; \mathbb{R})) \cap C^1([0, \infty); H^1(\mathbb{R}; \mathbb{R})),$$
$$A_T = \{\psi \in X;\ \operatorname{supp} \psi \subset (-\infty, T) \times \mathbb{R}\}.$$

Let $(\cdot \mid \cdot)$ be the usual $L^2(\mathbb{R})$ scalar product defined by $(f \mid g) = \int_{\mathbb{R}} f \overline{g}$. Then we define weak time-local solutions to (SR) as follows:

Definition 1.5.3. *Let $F(z)=\lambda|z|^p$ with $\lambda\in\mathbb{C}\backslash\{0\}$ and $p>1$. Let $T>0$ and $u_0\in L^1_{\mathrm{loc}}(\mathbb{R})$. We say that u is a time-local weak solution to* (SR)*, if u belongs to $L^1_{\mathrm{loc}}([0,T);L^p(\mathbb{R}))$ and the following identity*

$$\int_0^T \big(u(t)\big|i\partial_t\psi(t)\pm(m^2-\Delta)^{1/2}\psi(t)\big)dt=i(u_0|\psi(0))+\lambda\int_0^T\big(|u(t)|^p\big|\psi(t)\big)dt$$

holds for any $\psi\in A_T$, where the double-sign corresponds to the sign of (SR).

With this weak time-local solutions, we have the following:

Theorem 3. *Let $F(z)=\lambda|z|^p$ with $\lambda\in\mathbb{C}\backslash\{0\}$ and $p>1$. Then for any $f\in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfying*

$$\begin{aligned}&\exists\delta>0 \text{ s.t. } f>0 \text{ on } (-\delta,\delta),\\ &f \text{ is decreasing on } (0,\delta),\\ &\lim_{\varepsilon\searrow 0} f(\varepsilon)=\infty,\end{aligned}$$

there exists no $T>0$ such that there exists a local weak solution to (SR) *with $u_0=-i\overline{\lambda}f$.*

Remark 1.5.4. *We remark that*

$$f(x)=\sum_{m=1}^{\infty}\frac{1}{m}e^{-4^m x^2}\cos(2^m x)$$

belongs to $H^{1/2}(\mathbb{R})$ and satisfies the condition of Theorem 3. For the details of the character of f, we refer the reader to [84]. This implies that Theorem 3 also asserts that for some $H^{1/2}(\mathbb{R})$ initial data, there exist no weak time-local solutions to (SR) *with $F=\lambda|z|^p$. We also remark that for $s>1/2$, there exists no $H^s(\mathbb{R})$ function with a singularity at the origin, since $H^s(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$.*

If the Duhamel term of semirelativistic equation has smoothness property like that of Schrödinger or Klein-Gordon equations, then even in the $H^s(\mathbb{R})$ setting with $s_{1,p}^{(SR)}<s\le 1/2$, we have time-local solution to (SR) for any $H^s(\mathbb{R})$ initial data. Theorem 3 also implies that at least in the case of $\mathbb{R}$, the Duhamel term doesn't have a similar smoothness property to construct time-local solution.

Theorem 3 follows from a test function method which is introduced by Zhang in [93, 94]. To apply the test function method, we deform (SR) in order to cancel the non-locality of $(m^2-\Delta)^{1/2}$, since pointwise estimates of test functions are necessary for test function methods. Moreover, we use a special sequence of test functions introduced by a study of the non-existence of solutions of an advection equations so that we obtain the non-existence of weak time-local solutions to (SR) in scaling subcritical case.

1.6 Outline

At the end of this chapter, we give a brief outline of this thesis. In Chapter 2, we give an explanation of Fourier restriction method and proof of Theorem 1. In Chapter 3, we prove a priori estimates for charge and energy of solutions to (SR) and we prove Theorem 2. In Chapter 4, we explain how to deform (SR) in order to cancel $(m^2-\Delta)^{1/2}$ and show Theorem 3 with a special sequence of test functions. In Appendix A.1, we study the semirelativistic system by revisiting Fourier restriction method and a priori estimate. In Appendix A.2, we give an simple proof of Lemma 1.3.11 from the view point of weighted integral inequality and discuss about the condition of weights so that the associated integral inequality holds. In Appendix A.3, we show the sharp bilinear estimate of $H^s(\mathbb{R}^n)$ norm, fractional Leibniz rule, from the view point of Fourier multiplier.

Chapter 2

Well-Posedness of (SR) with Quadratic Nonlinearity

2.1 Introduction

In this chapter, we study the following Cauchy problems of semirelativistic equations:

$$\begin{cases} i\partial_t u \pm (m^2-\Delta)^{1/2}u = \lambda \overline{u}^2, & t\in[0,T], \quad x\in\mathbb{R}, \\ u(0)=u_0, & x\in\mathbb{R}, \end{cases} \tag{2.1.1}$$

$$\begin{cases} i\partial_t u \pm (m^2-\Delta)^{1/2}u = \lambda |u|^2, & t\in[0,T], \quad x\in\mathbb{R}, \\ u(0)=u_0, & x\in\mathbb{R}, \end{cases} \tag{2.1.2}$$

$$\begin{cases} i\partial_t u \pm (m^2-\Delta)^{1/2}u = \lambda u^2, & t\in[0,T], \quad x\in\mathbb{R}, \\ u(0)=u_0, & x\in\mathbb{R} \end{cases} \tag{2.1.3}$$

with $\lambda\in\mathbb{C}\backslash\{0\}$.

The purpose of this chapter is to show the criteria of the order of the Sobolev spaces with which each time-local solutions of (2.1.1), (2.1.2), and (2.1.3) can be constructed by contraction argument.

To motivate our problem, we revisit the earlier works: [13, 15, 62, 73]. Borgna and Rial studied the Cauchy problem for a single semirelativistic equation with cubic nonlinearity in [15] and they proved the existence of time-global solutions in the $H^s(\mathbb{R})$ setting with $s>1/2$. The method of their proof depends essentially on the Sobolev embedding $H^s(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$. In the case where $s\le 1/2$, however, the method loses its meaning because the uniform control by H^s norm breaks down. In the limiting case $s=1/2$, a Vladimirov type argument [76, 77, 92] implies that the uniqueness of weak solutions constructed by a compactness argument, see [62]. Meanwhile, Strichartz type estimates are known to be sharper linear estimate for Duhamel term but we remark that Strichartz type estimates are not sufficient for a contraction argument unless the uniform control by H^s norm is available. A similar situation happens in the case of nonlinear Dirac equations in space dimensions $n\ge 2$ [14, 30, 70, 71, 72]. We neither can not apply the Delgado-Candy trick which is the special technique for the Dirac equation in one dimension. This technique depends on algebraic structure of the Dirac equation to divided solutions into free solution part and uniform bounded part. However, the semirelativistic equation does not have the algebraic structure. See [13, 73].

From the view point of scaling criticality, (2.1.1), (2.1.2), and (2.1.3) are expected to be time-locally well-posed in the $H^s(\mathbb{R})$ setting with $s>s_{1,2}^{(SR)}=-1/2$. By focusing on the structure of nonlinearity, we have the following results:

Theorem 2.1.1. (2.1.1) *is time-locally well-posed if* $s \geq 0$. *Moreover,* $T(s) = T(0)$, *where* $T(s)$ *is the maximal existence time of solutions defined by*

$$T(s) = T(s, u_0) = \sup\big\{T > 0 \mid \sup_{0<t<T} \|u(t) : H^s\| < \infty\big\}.$$

Theorem 2.1.2. *The solution map of* (2.1.1) *is not* C^3 *in the* $H^s(\mathbb{R})$ *setting with* $-1/2 < s < 0$. *In particular, if* $-1/2 < s < 0$, *then for initial data*

$$u_0 = \mathfrak{F}^{-1}[\chi_{[1,\infty)}\langle\cdot\rangle^{-1/2-s-\varepsilon}] \in H^s(\mathbb{R})$$

with $0 < \varepsilon < -2s/3$,

$$\int_0^t U(\pm(t-t'))\left(\overline{U(\pm t')u_0}\int_0^{t'} U(\pm((t''-t'))(U(\pm t'')u_0)^2 dt''\right) dt' \notin H^s(\mathbb{R}). \tag{2.1.4}$$

Theorem 2.1.3. *The solution map of* (2.1.1) *is not* C^2 *in the* $H^s(\mathbb{R})$ *setting with* $s < -1/2$. *In particular, if* $s < -1/2$, *then for a sequence of initial datum* $u_{0,k}$ *defined by*

$$u_{0,k} = k^{-s}\mathfrak{F}^{-1}[\chi_{[-1,1]}(\cdot - k) + \chi_{[-1,1]}(\cdot + k)],$$

then there exists $C > 0$ *such that for any* k, $\|u_{0,k}\|_{H^s(\mathbb{R})} \leq C$ *and*

$$\limsup_{k\to\infty} \left\| \int_0^t U(\pm(t-t'))\overline{U(\pm t')u_{0,k}}^2 dt' \right\|_{H^s(\mathbb{R})} = \infty. \tag{2.1.5}$$

Theorem 2.1.4. *The solution maps of* (2.1.2) *and* (2.1.3) *are not* C^2 *in the* $H^s(\mathbb{R})$ *setting with* $-1/2 < s < 1/2$. *In particular, if* $-1/2 < s < 1/2$, *then for initial data*

$$u_0 = \mathfrak{F}^{-1}[\chi_{[0,\infty)}\langle\cdot\rangle^{-1/2-s-\varepsilon}] \in H^s(\mathbb{R})$$

with $0 < \varepsilon < 1/6 - s/3$,

$$\int_0^t U(\pm(t-t'))\big(\overline{U(\pm t')u_0}U(\pm t')u_0\big)dt' \notin H^s(\mathbb{R}), \tag{2.1.6}$$

$$\int_0^t U(\pm(t-t'))\big((U(\pm t')u_0)^2\big)dt' \notin H^s(\mathbb{R}). \tag{2.1.7}$$

Theorem 2.1.5. *The solution map of* (2.1.1) *is not* C^2 *in the* $H^s(\mathbb{R})$ *setting with* $s < -1/2$. *In particular, if* $s < -1/2$, *then for a sequence of initial datum* $u_{0,k}$ *defined by*

$$u_{0,k} = k^{-s}\mathfrak{F}^{-1}(\chi_{[-1,1]}(\xi - k))$$

then there exists $C > 0$ *such that for any* k, $\|u_{0,k}\|_{H^s} \leq C$ *and*

$$\limsup_{k\to\infty} \left\| \int_0^t U(\pm(t-t'))\big(\overline{U(\pm t')u_0}U(\pm t')u_0\big)dt' \right\|_{H^s(\mathbb{R})} = \infty,$$

$$\limsup_{k\to\infty} \left\| \int_0^t U(\pm(t-t'))\big((U(\pm t')u_0)^2\big)dt' \right\|_{H^s(\mathbb{R})} = \infty.$$

The well-posedness of (2.1.1) follows from Fourier restriction method which is introduced Bourgain in [9, 10, 11]. It is natural to introduce Fourier restriction method to study (2.1.1),(2.1.2), and (2.1.3) in the H^s setting with $0 \leq s \leq 1/2$. It is because Strichartz type estimates are not sufficient to study (2.1.1),(2.1.2), and (2.1.3) and therefore it seems difficult to improve linear estimates for their Duhamel terms. In addition, the charge and energy of solutions to (2.1.1),(2.1.2), and (2.1.3) are not conserved. Therefore, we don't expect compactness argument helpful to study them. The Fourier restriction method is a method to study (SR) with Fourier restriction norms defined below. We can estimate nonlinearity sharply with Fourier restriction norm, since Fourier restriction norm is a L^2 norm weighted by Fourier multipliers with regard to the main part of the Cauchy problem. In particular, one can easily see the nonlinear interaction of each of frequencies of time and space by these Fourier multipliers. However, in the $L^2(\mathbb{R})$ setting, we see that it seems difficult to construct time-local solutions to (2.1.1) based only on the Fourier restriction norm, by showing the failure of the corresponding nonlinear estimates. Therefore, we also introduce auxiliary norms below.

The non-smoothness of solution maps follows from direct calculations of Duhamel term with linear solution. We remark that Theorem 2.1.5 is shown by a similar method to the proof of Theorem 2.1.3, so we omit the proof of Theorem 2.1.5.

The rest of this chapter is organized as follows. In section 2.2, we give notation and collect basic facts. In section 2.3, we show Theorem 2.1.1 with Fourier restriction method. In section 2.4, we show Theorem 2.1.2. In section 2.5, we show Theorem 2.1.3. In section 2.6, we show Theorem 2.1.4.

2.2 Preliminary

2.2.1 Notation

For $u : \mathbb{R}^2 \ni (t,x) \mapsto u(t,x) \in \mathbb{C}$, let

$$\widetilde{u}(\tau,\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}(t,\xi) \exp(-it\tau) dt.$$

For $m \geq 0$, $s, b \in \mathbb{R}$, $T_0 \in \mathbb{R}$, and $T > 0$, we define Fourier restriction norms as follows:

$$\|u\|_{X^{s,b}_{m,\pm}} = \|\langle \xi \rangle^s \langle \tau \pm \sqrt{m^2+\xi^2} \rangle^b \widetilde{u}\|_{L^2(\mathbb{R}\times\mathbb{R})},$$

$$\|u\|_{X^{s,b}_{m,\pm}[T_0,T_0+T]} = \inf \left\{ \|u'\|_{X^{s,b}_{m,\pm}} \;\middle|\; \begin{array}{l} u'(t,x) = u(t,x) \text{ on } [T_0, T_0+T] \times \mathbb{R}, \\ \text{supp } u' \subset [T_0 - 2T, T_0 + 2T] \times \mathbb{R} \end{array} \right\},$$

$$\|u\|_{X'^{s,b}_{m,\pm}[T_0,T_0+T]} = \inf \left\{ \|u'\|_{X^{s,b}_{m,\pm}} \;\middle|\; u'(t,x) = u(t,x) \text{ on } [T_0, T_0+T] \times \mathbb{R} \right\}.$$

We also define auxiliary norms as follows:

$$\|u\|_{Y^{s}_{m,\pm}} = \|\langle \xi \rangle^s \langle \tau \pm \sqrt{m^2+\xi^2} \rangle^{-1} \widetilde{u}\|_{L^2(\mathbb{R}_\xi; L^1(\mathbb{R}_\tau))},$$

$$\|u\|_{Y^{s}_{m,\pm}[T_0,T_0+T]} = \inf \left\{ \|u'\|_{Y^{s}_{m,\pm}} \;\middle|\; \begin{array}{l} u'(t,x) = u(t,x) \text{ on } [T_0, T_0+T] \times \mathbb{R}, \\ \text{supp } u' \subset [T_0 - 2T, T_0 + 2T] \times \mathbb{R} \end{array} \right\},$$

where

$$\|f(\tau,\xi)\|_{L^2(\mathbb{R}_\xi;L^1(\mathbb{R}_\tau))} = \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f(\tau,\xi)|d\tau\right)^2 d\xi\right)^{1/2}.$$

We define a norm space $X^{s,b}_{m,\pm}$ as follows:

$$X^{s,b}_{m,\pm} = \{u \in L^2(\mathbb{R}\times\mathbb{R}) \mid \|u\|_{X^{s,b}_{m,\pm}} < \infty\}.$$

We also define $X^{s,b}_{m,\pm}[T_0,T_0+T]$, $X'^{s,b}_{m,\pm}[T_0,T_0+T]$, $Y^s_{m,\pm}$, and $Y^s_{m,\pm}[T_0,T_0+T]$ similarly. We abbreviate these spaces as : $X^{s,b}_{\pm} = X^{s,b}_{0,\pm}$, $Y^s_{\pm} = Y^s_{0,\pm}$.

Let ψ be a smooth function with $0 \le \psi \le 1$, $\psi(t) = 1$ if $|t| \le 1$ and $\psi(t) = 0$ if $|t| \ge 2$. For $T > 0$, let $\psi_T(t) = \psi(T^{-1}t)$.

2.2.2 Basic Characteristics of Fourier Restriction Norms

Here we collect some basic facts of Fourier restriction norms and auxiliary norms.

Lemma 2.2.1. *For any $s,b \ge 0$, $T > 0$ and $m, T_0 \in \mathbb{R}$, $X^{s,b}_{m,\pm}$ and $X^{s,b}_{m,\pm}[T_0,T_0+T]$ are Banach spaces.*

proof. For any $s,b \ge 0$, and $m \in \mathbb{R}$,

$$\begin{aligned}&\langle\xi\rangle^{-s}\langle\tau \pm \sqrt{m^2+\xi^2}\rangle^{-b}L^2(\mathbb{R}\times\mathbb{R})\\ &= \{f \in L^2(\mathbb{R}\times\mathbb{R}) \mid \|f\|_{\langle\xi\rangle^{-s}\langle\tau\pm\sqrt{m^2+\xi^2}\rangle^{-b}L^2(\mathbb{R}\times\mathbb{R})} < \infty\}\end{aligned}$$

is a Banach space, where

$$\|f\|_{\langle\xi\rangle^{-s}\langle\tau\pm\sqrt{m^2+\xi^2}\rangle^{-b}L^2(\mathbb{R}\times\mathbb{R})} = \|\langle\xi\rangle^{s}\langle\tau \pm \sqrt{m^2+\xi^2}\rangle^{b} f\|_{L^2(\mathbb{R}\times\mathbb{R})}.$$

Indeed, let $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $\langle\xi\rangle^{-s}\langle\tau\pm\sqrt{m^2+\xi^2}\rangle^{-b}L^2(\mathbb{R}\times\mathbb{R})$, then $(\langle\xi\rangle^{s}\langle\tau\pm\sqrt{m^2+\xi^2}\rangle^{b}f_n)_{n\in\mathbb{N}}$ converges a L^2 function F in $L^2(\mathbb{R}\times\mathbb{R})$. Then $\langle\xi\rangle^{-s}\langle\tau \pm \sqrt{m^2+\xi^2}\rangle^{-b}F$ is the limit of $(f_j)_{j\in\mathbb{N}}$ in $\langle\xi\rangle^{-s}\langle\tau \pm \sqrt{m^2+\xi^2}\rangle^{-b}L^2(\mathbb{R}\times\mathbb{R})$. Since $X^{s,b}_{m,\pm}$ is the set of the inverse Fourier transformations of all elements of $\langle\xi\rangle^{-s}\langle\tau \pm \sqrt{m^2+\xi^2}\rangle^{-b}L^2(\mathbb{R}\times\mathbb{R})$, $X^{s,b}_{m,\pm}$ is also a Banach space. On the other hand,

$$B = \{f \in L^2(\mathbb{R}\times\mathbb{R}) \mid \operatorname{supp} f \subset [T_0-2T, T_0+2T]\times\mathbb{R}\}$$

is also shown to be a closed subspace of $L^2(\mathbb{R}\times\mathbb{R})$ as follows. Let $(f_j)_{j\in\mathbb{N}} \subset B$, $f \in L^2(\mathbb{R}\times\mathbb{R})$ be the limit of $(f_j)_{j\in\mathbb{N}}$. Let $\zeta \in C^\infty(\mathbb{R}\times\mathbb{R})$ satisfy $\zeta = 0$ on $[T_0-2T, T_0+2T]\times\mathbb{R}$ and $\zeta > 0$ on $[T_0-2T, T_0+2T]^c\times\mathbb{R}$. Then for any $\zeta' \in C^\infty(\mathbb{R}\times\mathbb{R})$ satisfying $\operatorname{supp}\zeta' \subset [T_0-2T, T_0+2T]^c\times\mathbb{R}$,

$$\begin{aligned}\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\zeta f\zeta'\,dt\,dx\right| &= \lim_{n\to\infty}\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\zeta(f-f_n)\zeta'\,dt\,dx\right|\\ &\le \lim_{n\to\infty}\|\zeta'\|_{L^2(\mathbb{R}\times\mathbb{R})}\|f-f_n\|_{L^2(\mathbb{R}\times\mathbb{R})}\\ &= 0.\end{aligned}$$

Therefore, $f \in B$ and B is closed. Similarly

$$M = \{f \in L^2(\mathbb{R}\times\mathbb{R}) \mid \operatorname{supp} f \subset [T_0, T_0+T]^c\times\mathbb{R}\}$$

is also closed subspace of $L^2(\mathbb{R}\times\mathbb{R})$. Then, since $X^{s,b}_{m,\pm} \hookrightarrow L^2(\mathbb{R}\times\mathbb{R})$, $(B \cap X^{s,b}_{m,\pm}\ ;\ \|\cdot\|_{X^{s,b}_{m,\pm}})$ is also a Banach space and $B \cap M \cap X^{s,b}_{m,\pm}$ is a close subspace of $(B \cap X^{s,b}_{m,\pm})$. By Lemma 1.3.2, $X^{s,b}_{m,\pm}[T_0,T_0+T] = (B\cap X^{s,b}_{m,\pm})\backslash(B\cap M\cap X^{s,b}_{m,\pm})$ is also a Banach space. Q.E.D.

Lemma 2.2.2. *If $u \in X^{s,b}_{m,\pm}$, then $\overline{u} \in X^{s,b}_{m,\mp}$.*

***proof*.** Since

$$\mathfrak{F}[\overline{f}](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \overline{f} e^{ix\xi} dx = \frac{1}{\sqrt{2\pi}} \overline{\int_{\mathbb{R}} f e^{-ix\xi} dx} = \overline{\mathfrak{F}f(-\xi)}$$

we are done. Q.E.D.

Due to the next lemma, we may put $m = 0$ with respect to the Fourier restriction and auxiliary norms without loss of generality.

Lemma 2.2.3. *For any $m, M \in \mathbb{R}$, $X^{s,b}_{m,\pm} \simeq X^{s,b}_{M,\pm}$, $Y^s_{m,\pm} \simeq Y^s_{M,\pm}$ with equivalent norms.*

***proof*.** The lemma follows from the following inequality:

$$\begin{aligned} \frac{\langle \tau \pm \sqrt{m^2+\xi^2} \rangle}{\langle \tau \pm \sqrt{M^2+\xi^2} \rangle} &\le 1 + \left| \frac{\langle \tau + \sqrt{m^2+\xi^2} \rangle - \langle \tau + \sqrt{M^2+\xi^2} \rangle}{\langle \tau \pm \sqrt{M^2+\xi^2} \rangle} \right| \\ &= 1 + \frac{\left| |\tau \pm \sqrt{m^2+\xi^2}| - |\tau \pm \sqrt{M^2+\xi^2}| \right|}{\langle \tau \pm \sqrt{M^2+\xi^2} \rangle} \\ &\le 1 + |m - M| \end{aligned}$$

for any $\xi, \tau \in \mathbb{R}$. Q.E.D.

2.2.3 Linear Estimates

Here we collect some linear estimates with $X^{s,b}_{m,\pm}$ and $Y^s_{m,\pm}$.

Lemma 2.2.4 ([36, (2.19)])**.** *Let $m \in \mathbb{R}$. For any $s, b \ge 0$ and $u_0 \in H^s$,*

$$\|\psi(t)U(\pm t)u_0\|_{X^{s,b}_{m,\mp}} = \|\psi\|_{H^b} \|u_0\|_{H^s}. \tag{2.2.1}$$

In addition, for any $0 < T < 1$,

$$\|\psi_T(t)U(\pm t)u_0\|_{X^{s,1/2}_{m,\mp}} \lesssim \|u_0\|_{H^s}. \tag{2.2.2}$$

***proof*.** The equality (2.2.1) follows from

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-it\tau)\psi(t)\mathfrak{F}[U(\pm t)u_0] dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-it(\tau \mp \sqrt{m^2+\xi^2}))\psi(t)\hat{u}_0(\xi) dt \\ &= \hat{\psi}(\tau \mp \sqrt{m^2+\xi^2})\hat{u}_0(\xi). \end{aligned}$$

The estimate (2.2.2) follows form

$$\begin{aligned} \|\psi_T\|_{H^{1/2}(\mathbb{R})} &\le \|\psi_T\|_{L^2(\mathbb{R})} + \|\psi_T\|_{\dot{H}^{1/2}(\mathbb{R})} \\ &= T^{1/2}\|\psi\|_{L^2(\mathbb{R})} + \|\psi\|_{\dot{H}^{1/2}(\mathbb{R})} \\ &\le \|\psi\|_{H^{1/2}(\mathbb{R})}. \end{aligned}$$

Q.E.D.

Proposition 2.2.1 ([36, Lemma 2.1.]). *Let $m \in \mathbb{R}$, $0 < T \leq 1$ and let $s \geq 0$. Then,*

$$\left\| \psi_T(t) \int_0^t U(\pm(t-t'))F(t')\ dt' \right\|_{X^{s,1/2}_{m,\mp}} \lesssim \|F\|_{X^{s,-1/2}_{m,\mp} \cap Y^s_{m,\mp}}$$

for $F \in X^{s,-1/2}_{m,\mp} \cap Y^s_{m,\mp}$. In addition, let $\delta \geq 0$ and b satisfy $-1/2 < b - 1 + \delta \leq 0 \leq b$. Then,

$$\left\| \psi_T(t) \int_0^t U(\pm(t-t'))F(t')\ dt' \right\|_{X^{s,b}_{m,\mp}} \lesssim T^\delta \|F\|_{X^{s,b-1+\delta}_{m,\mp}}$$

for $F \in X^{s,b-1+\delta}_{m,\mp}$.

Lemma 2.2.5 ([36, Lemma 2.2.]). *Let $m \in \mathbb{R}$. If $F \in Y^s_{m,\pm}$, then $\int_0^\cdot U_m(\cdot - t')F(t')dt' \in C(\mathbb{R}; H^s)$ and it satisfies the estimate*

$$\left\| \int_0^\cdot U(\pm(\cdot - t'))F(t')dt' \right\|_{C(\mathbb{R}:H^s)} \lesssim \|F\|_{Y^s_{m,\mp}}.$$

To extract a positive power of T, we use the following lemma.

Lemma 2.2.6 ([36, Lemma 3.1.]). *Let $s \in \mathbb{R}$, $0 \leq b \leq b'$, $T > 0$ and let $f \in X^{s,b'}_{\pm}$ satisfy* $\mathrm{supp} f \subset [-T, T] \times \mathbb{R}$. *Then,*

$$\|f\|_{X^{s,b}_{\pm}} \lesssim T^{\gamma(b',b)} \|f\|_{X^{s,b'}_{\pm}},$$

where

$$\gamma(b',b) = \begin{cases} b' - b & \text{if} \quad b' < 1/2, \\ b' - b + \varepsilon & \text{if} \quad b' = 1/2, \\ 1/2 - b/2b' & \text{if} \quad b' > 1/2 \end{cases}$$

with $\varepsilon > 0$ sufficiently small.

proof. By the Hölder inequality,

$$\begin{aligned} &\|\langle\xi\rangle^s \langle\tau \pm |\xi|\rangle^b \widetilde{f}\|_{L^2(\mathbb{R}\times\mathbb{R})} \\ &\leq \left\| \left\{ \langle\xi\rangle^s \langle\tau \pm |\xi|\rangle^{b'} \widetilde{f} \right\}^{b/b'} \right\|_{L^{2b'/b}(\mathbb{R}\times\mathbb{R})} \left\| \left\{ \langle\xi\rangle^s \widetilde{f} \right\}^{1-b/b'} \right\|_{L^{2b'/(b'-b)}(\mathbb{R}\times\mathbb{R})} \\ &= \|f\|^{b/b'}_{X^{s,b'}_{\pm}} \|\langle\xi\rangle^s \hat{f}\|_{L^2([-T,T]\times\mathbb{R})}. \end{aligned}$$

If $b' > 1/2$, then

$$\begin{aligned} \|\langle\xi\rangle^s \hat{f}\|_{L^2([-T,T]\times\mathbb{R})} &\lesssim T^{1/2} \|\langle\xi\rangle^s \hat{f}\|_{L^2(\mathbb{R}_\xi; L^\infty(\mathbb{R}_t))} \\ &\leq T^{1/2} \|\langle\xi\rangle^s \widetilde{f}\|_{L^2(\mathbb{R}_\xi; L^1(\mathbb{R}_\tau))} \\ &\lesssim T^{1/2} \|f\|_{X^{s,b'}_{\pm}}. \end{aligned}$$

Moreover, if $b' < 1/2$, then by the unitarity of U and the Sobolev embedding,

$$\begin{aligned} \|\langle\xi\rangle^s \mathfrak{F}_x[f]\|_{L^2(\mathbb{R}\times\mathbb{R})} &= \|\langle\xi\rangle^s \mathfrak{F}_x[U(\pm t)f]\|_{L^2(\mathbb{R}\times\mathbb{R})} \\ &\lesssim T^{b'} \|\langle\xi\rangle^s \mathfrak{F}_x[U(\pm t)f]\|_{L^2(\mathbb{R}_\xi; L^{2/(1-2b')}(\mathbb{R}_t))} \\ &\lesssim T^{b'} \|\langle\xi\rangle^s \mathfrak{F}_x[U(\pm t)f\|_{L^2(\mathbb{R}_\xi; H^{b'}(\mathbb{R}_t))} \\ &= T^{b'} \|f\|_{X^{s,b'}_{\pm}}. \end{aligned}$$

In the case $b' = 1/2$, for any $\varepsilon > 0$,

$$\|\langle\xi\rangle^s \mathfrak{F}_x[f]\|_{L^2(\mathbb{R}\times\mathbb{R})} \lesssim T^{1/2-\varepsilon}\|f\|_{X_\pm^{s,1/2-\varepsilon}} \leq T^{1/2-\varepsilon}\|f\|_{X_\pm^{s,1/2}}$$

Q.E.D.

2.2.4 Bilinear and Trilinear Estimates

In this section, we derive nonlinear estimates for $X^{s,b}_{m,\pm}$ and $Y^s_{m,\pm}$ by the method originally proposed in [86].

Lemma 2.2.7. *Let $p \geq 1$ and let $\alpha,\ \beta,\ \gamma \geq 0$ satisfy $\alpha+\beta+\gamma > 1/p$. Then, there exists a positive constant C such that the inequality*

$$\|\langle\cdot+\delta_1\rangle^{-\alpha} f * g\|_{L^p(\mathbb{R})} \leq \|\langle\cdot+\delta_2\rangle^{\beta} f\|_{L^2(\mathbb{R})}\|\langle\cdot+\delta_3\rangle^{\gamma} g\|_{L^2(\mathbb{R})}$$

holds for any real numbers $\delta_1,\ \delta_2,\ \delta_3$ and any $f,\ g$ such that all the norms on the right hand side are finite.

***proof*.** By the Hölder and Young inequalities,

$$\begin{aligned}
&\|\langle\cdot+\delta_1\rangle^{-\alpha} f * g\|_{L^p(\mathbb{R})}\\
&\leq \|\langle\cdot\rangle^{-\alpha}\|_{L^{p\frac{\alpha+\beta+\gamma}{\alpha}}(\mathbb{R})}\|f*g\|_{L^{p\frac{\alpha+\beta+\gamma}{\beta+\gamma}}(\mathbb{R})}\\
&\leq \|\langle\cdot\rangle^{-1}\|^{\alpha}_{L^{p(\alpha+\beta+\gamma)}(\mathbb{R})}\|f\|_{L^{\frac{1}{\frac{1}{2}+\frac{\beta}{p(\alpha+\beta+\gamma)}}}(\mathbb{R})}\|g\|_{L^{\frac{1}{\frac{1}{2}+\frac{\gamma}{p(\alpha+\beta+\gamma)}}}(\mathbb{R})}\\
&\leq \|\langle\cdot\rangle^{-1}\|^{\alpha+\beta+\gamma}_{L^{p(\alpha+\beta+\gamma)}(\mathbb{R})}\|\langle\cdot+\delta_2\rangle^{\beta} f\|_{L^2(\mathbb{R})}\|\langle\cdot+\delta_3\rangle^{\gamma} g\|_{L^2(\mathbb{R})},
\end{aligned}$$

where if the denominator of an exponent is 0, we interpret the exponent as ∞. Then, we obtain the lemma. Q.E.D.

Lemma 2.2.8. *Let p and α satisfy $p \geq 1$ and $0 \leq \alpha \leq 1/p$. Let β,γ,κ satisfy $0 \leq \beta,\gamma,\kappa \leq 1/2$ and $\alpha+\beta+\gamma+\kappa > 1/p+1/2$. Then, there exists a positive constant C such that the inequality*

$$\begin{aligned}
&\|\langle\cdot+\delta_1\rangle^{-\alpha} f * g * h\|_{L^p(\mathbb{R})}\\
&\leq C\|\langle\cdot+\delta_2\rangle^{\beta} f\|_{L^2(\mathbb{R})}\|\langle\cdot+\delta_3\rangle^{\gamma} g\|_{L^2(\mathbb{R})}\|\langle\cdot+\delta_4\rangle^{\kappa} h\|_{L^2(\mathbb{R})}
\end{aligned}$$

holds for any real numbers $\delta_1,\delta_2,\delta_3,\delta_4$ and any $f,\ g,\ h$ such that all the norms on the right hand side are finite.

***proof*.** Let $\varepsilon = \alpha+\beta+\gamma+\kappa-1/p-1/2$. By the Hölder and the Young inequalities,

$$\begin{aligned}
&\|\langle\cdot+\delta_1\rangle^{-\alpha} f * g * h\|_{L^p(\mathbb{R})}\\
&\lesssim \|f*g*h\|_{L^{p_1}(\mathbb{R})}\\
&\lesssim \|f\|_{L^{p_2}(\mathbb{R})}\|g*h\|_{L^{p_3}(\mathbb{R})}\\
&\lesssim \|f\|_{L^{p_2}}\|g\|_{L^{p_4}(\mathbb{R})}\|h\|_{L^{p_5}(\mathbb{R})}\\
&\lesssim \|\langle\cdot+\delta_2\rangle^{\beta} f\|_{L^2(\mathbb{R})}\|\langle\cdot+\delta_3\rangle^{\gamma} g\|_{L^2(\mathbb{R})}\|\langle\cdot+\delta_4\rangle^{\kappa} h\|_{L^2(\mathbb{R})},
\end{aligned}$$

where

$$\begin{aligned}\frac{1}{p_1}&=\frac{1}{p}-\alpha+\frac{\alpha\varepsilon}{\alpha+\beta+\gamma+\kappa},\\ \frac{1}{p_2}&=\frac{1}{2}+\beta-\frac{\beta\varepsilon}{\alpha+\beta+\gamma+\kappa},\\ \frac{1}{p_3}&=\frac{1}{p_1}+1-\frac{1}{p_2},\\ \frac{1}{p_4}&=\frac{1}{2}+\gamma-\frac{\gamma\varepsilon}{\alpha+\beta+\gamma+\kappa},\\ \frac{1}{p_5}&=\frac{1}{2}+\kappa-\frac{\kappa\varepsilon}{\alpha+\beta+\gamma+\kappa}.\end{aligned}$$

Therefore, we obtain the lemma. Q.E.D.

For $s\geq 0$, we define $\lambda(s)$ as

$$\lambda(s)=\begin{cases}0 & \text{if} \quad s<1/2,\\ s-1/2+\varepsilon & \text{if} \quad s\geq 1/2,\end{cases} \tag{2.2.3}$$

where $\varepsilon>0$ is sufficiently small. Here we state our main nonlinear estimates.

Proposition 2.2.2. *Let $s\geq 0$ and $0\leq\rho<1/2$. Then, the inequality*

$$\|uv\|_{X_+^{s,-1/2}\cap Y_+^s}\lesssim\|u\|_{X_-^{\lambda(s),1/2}}\|v\|_{X_-^{s,1/2-\rho}}+\|u\|_{X_-^{\lambda(s),1/2-\rho}}\|v\|_{X_-^{s,1/2}} \tag{2.2.4}$$

holds for any $u\in X_-^{\lambda(s),1/2}$ and $v\in X_-^{s,1/2}$.

We remark that the regularity $\lambda(s)$ in the both terms of u on the right hand side is less than the regularity s on the left hand side. Therefore, the estimate (2.2.4) with $s>0$ does not follow directly from (2.2.4) with $s=0$ and the Peetre's inequality: $\langle\xi\rangle^{s'}\lesssim(\langle\xi-\eta\rangle^{s'}+\langle\eta\rangle^{s'})$ for $s'\geq 0$. We can exchange the smoothness with respect to the space-time variables into the smoothness with respect to the space variable by using (2.2.7) from the nice combination of signs in (2.2.4). This technique is found in Lemma 5 of [86].

The symmetry inequality

$$\|uv\|_{X_-^{s,-1/2}\cap Y_-^s}\lesssim\|u\|_{X_+^{\lambda(s),1/2}}\|v\|_{X_+^{s,1/2-\rho}}+\|u\|_{X_+^{\lambda(s),1/2-\rho}}\|v\|_{X_+^{s,1/2}}$$

holds by (2.2.4) with taking complex conjugate of u and v.

***proof*.** It is enough to show

$$\|uv\|_{X_+^{s,-1/2}}\lesssim\|u\|_{X_-^{\lambda(s),1/2}}\|v\|_{X_-^{s,1/2-\rho}}+\|u\|_{X_-^{\lambda(s),1/2-\rho}}\|v\|_{X_-^{s,1/2}} \tag{2.2.5}$$

and

$$\|uv\|_{Y_+^s}\lesssim\|u\|_{X_-^{\lambda(s),1/2}}\|v\|_{X_-^{s,1/2-\rho}}+\|u\|_{X_-^{\lambda(s),1/2-\rho}}\|v\|_{X_-^{s,1/2}}. \tag{2.2.6}$$

Let

$$M(\tau,\xi,\sigma,\eta)=\max\left(\big|\tau+|\xi|\big|,\big|\tau-\sigma-|\xi-\eta|\big|,\big|\sigma-|\eta|\big|\right).$$

Then, the triangle inequality implies

$$|\xi| + |\xi - \eta| + |\eta| \le 3M(\tau, \xi, \sigma, \eta). \tag{2.2.7}$$

Also, we decompose the integral region as follows:

$$\begin{aligned} A_1 &= \{(\tau, \xi, \sigma, \eta) \mid M(\tau, \xi, \sigma, \eta) = \big|\tau + |\xi|\big|\}, \\ A_2 &= \{(\tau, \xi, \sigma, \eta) \mid M(\tau, \xi, \sigma, \eta) = \big|\tau - \sigma - |\xi - \eta|\big|\}, \\ A_3 &= \{(\tau, \xi, \sigma, \eta) \mid M(\tau, \xi, \sigma, \eta) = \big|\sigma - |\eta|\big|\}. \end{aligned}$$

Then we show each of (2.2.5) and (2.2.6) in two different cases: where $s > 0$ and where $s = 0$.

(a) X norm estimate with $s > 0$.
By the Minkowski inequality,

$$\begin{aligned} &\left\| \langle \xi \rangle^s \iint_{\mathbb{R}^2} \langle \tau + |\xi| \rangle^{-1/2} \chi_{A_1}(\tau, \xi, \sigma, \eta)\ \widetilde{u}(\tau - \sigma, \xi - \eta)\ \widetilde{v}(\sigma, \eta)\ d\sigma d\eta \right\|_{L^2(\mathbb{R}_\tau \times \mathbb{R}_\xi)} \\ &\lesssim \left\| \int_{\mathbb{R}} \langle \cdot \rangle^{s-1/2} I_1(\cdot, \eta)\ d\eta \right\|_{L^2(\mathbb{R})}, \end{aligned}$$

where

$$I_1(\xi, \eta) = \left\| \int_{\mathbb{R}} \big|\widetilde{u}(\cdot - \sigma, \xi - \eta)\ \widetilde{v}(\sigma, \eta)\big|\ d\sigma \right\|_{L^2(\mathbb{R})}.$$

By Lemma 2.2.7

$$I_1(\xi, \eta) \lesssim \big\| \langle \cdot - |\xi - \eta| \rangle^{1/2-\rho} \widetilde{u}(\cdot, \xi - \eta) \big\|_{L^2(\mathbb{R})} \big\| \langle \cdot - |\eta| \rangle^{1/2} \widetilde{v}(\cdot, \eta) \big\|_{L^2(\mathbb{R})}.$$

Since

$$\begin{aligned} &\frac{1}{2} - s + \lambda(s) + s \ge \frac{1}{2}, \\ &\frac{1}{2} - s + \lambda(s) > 0, \end{aligned}$$

and Lemma 1.3.11,

$$\left\| \int_{\mathbb{R}} \langle \cdot \rangle^{s-1/2} I_1(\cdot, \eta)\ d\eta \right\|_{L^2(\mathbb{R})} \lesssim \|u\|_{X_-^{\lambda(s), 1/2-\rho}} \|v\|_{X_-^{s, 1/2}}.$$

Similarly, for $j = 2, 3$,

$$\begin{aligned} &\left\| \langle \xi \rangle^s \iint_{\mathbb{R}^2} \langle \tau + |\xi| \rangle^{-1/2} \chi_{A_j}(\tau, \xi, \sigma, \eta)\ \widetilde{u}(\tau - \sigma, \xi - \eta)\ \widetilde{v}(\sigma, \eta)\ d\sigma d\eta \right\|_{L^2(\mathbb{R}_\tau \times \mathbb{R}_\xi)} \\ &\lesssim \left\| \int_{\mathbb{R}} \langle \cdot \rangle^{s-1/2} I_j(\cdot, \eta)\ d\eta \right\|_{L^2(\mathbb{R})}, \end{aligned}$$

where

$$\begin{aligned} I_2(\xi, \eta) &= \left\| \langle \cdot + |\xi| \rangle^{-1/2} \int_{\mathbb{R}} \langle \cdot - \sigma - |\xi - \eta| \rangle^{1/2} \Big| \widetilde{u}(\cdot - \sigma, \xi - \eta)\ \widetilde{v}(\sigma, \eta) \Big|\ d\sigma \right\|_{L^2(\mathbb{R})}, \\ I_3(\xi, \eta) &= \left\| \langle \cdot + |\xi| \rangle^{-1/2} \int_{\mathbb{R}} \langle \sigma - |\eta| \rangle^{1/2} \Big| \widetilde{u}(\cdot - \sigma, \xi - \eta)\ \widetilde{v}(\sigma, \eta) \Big|\ d\sigma \right\|_{L^2(\mathbb{R})}. \end{aligned}$$

By Lemma 2.2.7,

$$\begin{aligned} I_2(\xi,\eta) &\lesssim \|\langle \cdot - |\xi-\eta|\rangle^{1/2}\widetilde{u}(\cdot,\xi-\eta)\|_{L^2(\mathbb{R})}\|\langle \cdot - |\eta|\rangle^{1/2-\rho}\widetilde{v}(\cdot,\eta)\|_{L^2(\mathbb{R})}, \\ I_3(\xi,\eta) &\lesssim \|\langle \cdot - |\xi-\eta|\rangle^{1/2-\rho}\widetilde{u}(\cdot,\xi-\eta)\|_{L^2(\mathbb{R})}\|\langle \cdot - |\eta|\rangle^{1/2}\widetilde{v}(\cdot,\eta)\|_{L^2(\mathbb{R})}. \end{aligned}$$

Then, for $j=2,3$, by Lemma 1.3.11

$$\begin{aligned} &\left\| \int_{\mathbb{R}} \langle \cdot \rangle^{s-1/2} I_j(\cdot,\eta) d\eta \right\|_{L^2(\mathbb{R})} \\ &\lesssim \|u\|_{X_-^{\lambda(s),1/2}}\|v\|_{X_-^{s,1/2-\rho}} + \|u\|_{X_-^{\lambda(s),1/2-\rho}}\|v\|_{X_-^{s,1/2}}. \end{aligned}$$

(b) X norm estimate with $s=0$.
By Lemmas 1.3.11 and 2.2.7,

$$\begin{aligned} &\|uv\|_{X_+^{0,-1/2}} \\ &\lesssim \sum_{j=1}^{3} \left\| \int_{\mathbb{R}} \langle \cdot \rangle^{-1/4}\langle \eta\rangle^{-1/4} I_j(\cdot,\eta)\ d\eta \right\|_{L^2(\mathbb{R})} \\ &\lesssim \|u\|_{X_-^{0,1/2}}\|v\|_{X_-^{0,1/2-\rho}} + \|u\|_{X_-^{0,1/2-\rho}}\|v\|_{X_-^{0,1/2}}, \end{aligned}$$

where I_1, I_2, and I_3 are defined as in the case (a).

(c) Y norm estimate with $s>0$.
By the Minkowski inequality,

$$\begin{aligned} &\left\| \langle \xi \rangle^s \iint_{\mathbb{R}^2} \langle \tau + |\xi|\rangle^{-1} \chi_{A_1}(\tau,\xi,\sigma,\eta)\widetilde{u}(\tau-\sigma,\xi-\eta)\widetilde{v}(\sigma,\eta)\ d\sigma d\eta \right\|_{L^2(\mathbb{R}_\xi;L^1(\mathbb{R}_\tau))} \\ &\lesssim \left\| \int_{\mathbb{R}} \langle \cdot \rangle^{s-1/2} J_1(\cdot,\eta)\ d\eta \right\|_{L^2(\mathbb{R})}, \end{aligned}$$

where

$$J_1(\xi,\eta) = \left\| \langle \cdot + |\xi|\rangle^{-1/2} \int_{\mathbb{R}} |\widetilde{u}(\cdot-\sigma,\xi-\eta)\widetilde{v}(\sigma,\eta)|\ d\sigma \right\|_{L^1(\mathbb{R})}.$$

By Lemma 2.2.7,

$$J_1(\xi,\eta) \lesssim \|\langle \cdot - |\xi-\eta|\rangle^{1/2-\rho}\widetilde{u}(\cdot,\xi-\eta)\|_{L^2(\mathbb{R})}\|\langle \cdot - |\eta|\rangle^{1/2}\widetilde{v}(\cdot,\eta)\|_{L^2(\mathbb{R})}.$$

Then, we obtain

$$\left\| \int \langle \cdot \rangle^{s-1/2} J_1(\cdot,\eta)\ d\eta \right\|_{L^2(\mathbb{R})} \lesssim \|u\|_{X_-^{\lambda(s),1/2-\rho}}\|v\|_{X_-^{s,1/2}}$$

by Lemma 1.3.11. Similarly, for $j=2,3$,

$$\begin{aligned} &\left\| \iint_{\mathbb{R}^2} \langle \tau + |\xi|\rangle^{-1} \chi_{A_j}(\tau,\xi,\sigma,\eta)\widetilde{u}(\tau-\sigma,\xi-\eta)\widetilde{v}(\sigma,\eta)\ d\sigma d\eta \right\|_{L^2(\mathbb{R}_\xi;L^1(\mathbb{R}_\tau))} \\ &\lesssim \left\| \int_{\mathbb{R}} \langle \cdot \rangle^{s-1/2} J_j(\cdot,\eta)\ d\eta \right\|_{L^2(\mathbb{R})}, \end{aligned}$$

where

$$J_2(\xi,\eta) = \left\| \langle \cdot + |\xi| \rangle^{-1} \int_{\mathbb{R}} \langle \cdot - \sigma - |\xi-\eta| \rangle^{1/2} \left| \widetilde{u}(\cdot - \sigma, \xi-\eta) \widetilde{v}(\sigma,\eta) \right| \, d\sigma \right\|_{L^1(\mathbb{R})},$$

$$J_3(\xi,\eta) = \left\| \langle \cdot + |\xi| \rangle^{-1} \int_{\mathbb{R}} \langle \sigma - |\eta| \rangle^{1/2} \left| \widetilde{u}(\cdot - \sigma, \xi-\eta) \widetilde{v}(\sigma,\eta) \right| \, d\sigma \right\|_{L^1(\mathbb{R})}.$$

By Lemma 2.2.7,

$$J_2(\xi,\eta) \lesssim \| \langle \cdot - |\xi-\eta| \rangle^{1/2} \widetilde{u}(\cdot, \xi-\eta) \|_{L^2(\mathbb{R})} \| \langle \cdot - |\eta| \rangle^{1/2-\rho} \widetilde{v}(\cdot,\eta) \|_{L^2(\mathbb{R})},$$
$$J_3(\xi,\eta) \lesssim \| \langle \cdot - |\xi-\eta| \rangle^{1/2-\rho} \widetilde{u}(\cdot, \xi-\eta) \|_{L^2(\mathbb{R})} \| \langle \cdot - |\eta| \rangle^{1/2} \widetilde{v}(\cdot,\eta) \|_{L^2(\mathbb{R})}.$$

Then,

$$\left\| \int_{\mathbb{R}} \langle \cdot \rangle^{s-1/2} J_j(\cdot,\eta) \, d\eta \right\|_{L^2(\mathbb{R})}$$
$$\lesssim \|u\|_{X_-^{\lambda(s),1/2}} \|v\|_{X_-^{s,1/2-\rho}} + \|u\|_{X_-^{\lambda(s),1/2-\rho}} \|v\|_{X_-^{s,1/2}}$$

follows from Lemma 1.3.11.

(d) Y norm estimate with $s=0$.
By Lemmas 1.3.11 and 2.2.7,

$$\|uv : Y_+^0\|$$
$$\lesssim \sum_{j=1}^{3} \left\| \int_{\mathbb{R}} \langle \cdot \rangle^{-1/4} \langle \eta \rangle^{-1/4} J_j(\cdot,\eta) \, d\eta \right\|_{L^2(\mathbb{R})}$$
$$\lesssim \|u\|_{X_-^{0,1/2}} \|v\|_{X_-^{0,1/2-\rho}} + \|u\|_{X_-^{0,1/2-\rho}} \|v\|_{X_-^{0,1/2}},$$

where J_1, J_2, and J_3 are defined as in the case (c).

Q.E.D.

Remark 2.2.3. *Proposition 2.2.2 is almost optimal. See Proposition 2.3.3 and Corollary 2.3.1.*

Remark 2.2.4. *The trick of exchanging smoothness is not applicable to the bilinear estimates $X_+^{s,b-1} \hookrightarrow X_+^{s,b} X_\pm^{s,b}$ and $X_-^{s,b-1} \hookrightarrow X_-^{s,b} X_\pm^{s,b}$ which one needs to use Fourier restriction method for* (2.2.4) *and* (2.2.5)*. In addition, the bilinear estimates $X_+^{s,b-1} \hookrightarrow X_+^{s,b} X_\pm^{s,b}$ and $X_-^{s,b-1} \hookrightarrow X_-^{s,b} X_\pm^{s,b}$ fail for $s \le 1/2$ and any $b \in \mathbb{R}$. For any $s \le 1/2$ and $b \in \mathbb{R}$, let $\widetilde{u}_\pm = \langle \tau \pm \xi \rangle^{-b-1} \langle \xi \rangle^{-s-1/2} \log \langle \xi \rangle^{-3/4}$. Then, $u_\pm \in X_\pm^{s,b}$ and*

$$\|u_+ u_\pm\|_{X_+^{s,b}} = \|u_- u_\pm\|_{X_-^{s,b}} = \infty.$$

These estimates are calculated as follows:

$$
\begin{aligned}
&\|u_+u_+\|_{X_+^{s,b}} \\
&= \left\| \langle\xi\rangle^s \langle\tau+|\xi|\rangle^{b-1} \iint_{\mathbb{R}^2} \langle\tau-\sigma+|\xi-\eta|\rangle^{-1} \langle\sigma+|\eta|\rangle^{-1} \langle\xi-\eta\rangle^{-s-1/2} \log\langle\xi-\eta\rangle^{-3/4} \right. \\
&\qquad\qquad \left. \cdot \langle\eta\rangle^{-s-1/2} \log\langle\eta\rangle^{-3/4} d\sigma d\eta \right\|_{L^2(\mathbb{R}_\tau\times\mathbb{R}_\xi)} \\
&\geq \left\| \langle\xi\rangle^s \langle\tau+\xi\rangle^{b-1} \int_0^\xi \int_{-\eta-1}^{-\eta+1} \langle\tau-\sigma+\xi-\eta\rangle^{-1} \langle\sigma+\eta\rangle^{-1} \langle\xi-\eta\rangle^{-s-1/2} \langle\eta\rangle^{-s-1/2} \right. \\
&\qquad\qquad \left. \cdot \log\langle\xi-\eta\rangle^{-3/4} \log\langle\eta\rangle^{-3/4} d\sigma d\eta \right\|_{L^2(\{(\tau,\xi)|\xi\geq 2,\ -1\leq\tau+\xi\leq+1)\}} \\
&\gtrsim \left\| \langle\cdot\rangle^{-1/2} \log\langle\cdot\rangle^{-3/4} \int_0^{\cdot} \langle\eta\rangle^{-1} \log\langle\eta\rangle^{-3/4} d\eta \right\|_{L^2(2,\infty)} \\
&\gtrsim \|\langle\cdot\rangle^{-1/2} \log\langle\cdot\rangle^{-1/2}\|_{L^2(2,\infty)} = \infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\|u_+u_-\|_{X_+^{s,b}} \\
&= \left\| \langle\xi\rangle^s \langle\tau+|\xi|\rangle^{b-1} \iint_{\mathbb{R}^2} \langle\tau-\sigma+|\xi-\eta|\rangle^{-1} \langle\sigma-|\eta|\rangle^{-1} \langle\xi-\eta\rangle^{-s-1/2} \langle\eta\rangle^{-s-1/2} \right. \\
&\qquad\qquad \left. \cdot \log\langle\xi-\eta\rangle^{-3/4} \log\langle\eta\rangle^{-3/4} d\sigma d\eta \right\|_{L^2(\mathbb{R}_\tau\times\mathbb{R}_\xi)} \\
&\geq \left\| \langle\xi\rangle^s \langle\tau+\xi\rangle^{b-1} \int_{-\xi}^0 \int_{\eta-1}^{\eta+1} \langle\tau-\sigma+\xi-\eta\rangle^{-1} \langle\sigma+\eta\rangle^{-1} \langle\xi-\eta\rangle^{-s-1/2} \langle\eta\rangle^{-s-1/2} \right. \\
&\qquad\qquad \left. \cdot \log\langle\xi-\eta\rangle^{-3/4} \log\langle\eta\rangle^{-3/4} d\sigma d\eta \right\|_{L^2(\{(\tau,\xi)|\xi\geq 2,\ -1\leq\tau+\xi\leq 1\})} \\
&\gtrsim \left\| \langle\cdot\rangle^{-1/2} \log\langle\cdot\rangle^{-3/4} \int_0^{\cdot} \langle\eta\rangle^{-1} \log\langle\eta\rangle^{-3/4} d\eta \right\|_{L^2(2,\infty)} = \infty,
\end{aligned}
$$

and the remainders are estimated similarly.

Corollary 2.2.1. *Let* $s \geq 0$, $0 \leq \rho < 1/2$ *and let* $T > 0$. *Then,*

$$\|uv\|_{X_\pm^{s,-1/2}\cap Y_\pm^s} \lesssim T^\rho \|u\|_{X_\mp^{\lambda(s),1/2}} \|v\|_{X_\mp^{s,1/2}} \tag{2.2.8}$$

for any $u \in X_\mp^{\lambda(s),1/2}$ *and* $v \in X_\mp^{s,1/2}$ *such that* $\mathrm{supp}\ u,\ \mathrm{supp}\ v \subset [-T,T]\times\mathbb{R}$.

proof. By Proposition 2.2.2 and Lemma 2.2.6, we obtain (2.2.8). Q.E.D.

2.3 Proof of Theorem 2.1.1

We separate the proof for the existence and for the persistence of regularity.

2.3.1 Proof of Existence of Solutions

Let $s \geq 0$, $u_0 \in H^s$ and let $0 < T \leq 1$. We define a solution map $\Phi_\pm$ as

$$\Phi_\pm(u)(t) = U(\pm t)u_0 - i\lambda \int_0^t U(\pm(t-t'))\, \overline{u(t')}^2 dt', \tag{2.3.1}$$

where double-sign corresponds (2.1.1). We also define a metric space

$$B^s_\pm(R, [0,T]) = \{u \in X^{s,1/2}_\pm[0,T] \mid \|u\|_{X^{s,1/2}_\pm[0,T]} \leq R\}$$

with metric

$$d^s_\pm(u_1,\ u_2) = \|u_1 - u_2\|_{X^{s,1/2}_\pm[0,T]}.$$

We see $(B^s_\pm(R,[0,T]), d^s_\pm)$ is a complete metric space for any $s \geq 0$. We prove that $\Phi_\pm$ is a contraction map on $B^s_\mp(R,[0,T])$ for sufficiently large R and sufficiently small T.

Let $u \in B^s_\mp(R,[0,T])$ and let $u' \in X^{s,1/2}_\mp$ satisfy

$$u' = u \quad \text{on} \quad [0,T]\times\mathbb{R}, \quad \text{supp}\, u' \subset [-2T, 2T]\times\mathbb{R}.$$

Then, $\Phi_\pm(u)$ is defined on $[0,T]\times\mathbb{R}$. Moreover,

$$\psi_T(t)\int_0^t U(\pm(t-t'))\, \overline{u'(t')}^2\ dt' = \int_0^t U(\pm(t-t'))\, \overline{u(t')}^2 dt'$$

on $[0,T]\times\mathbb{R}$ and their supports are contained in $[-2T,2T]\times\mathbb{R}$. Then,

$$\begin{aligned}
&\|\Phi_\pm(u)\|_{X^{s,1/2}_\mp[0,T]} \\
&\leq \|U(\pm t)u_0\|_{X^{s,1/2}_\mp[0,T]} + \left\|\lambda\int_0^t U(\pm(t-t'))\, \overline{u(t')}^2 dt'\right\|_{X^{s,1/2}_-[0,T]}.
\end{aligned}$$

By Lemma 2.2.4,

$$\|U(\pm t)\ u_0\|_{X^{s,1/2}_\mp[0,T]} \leq \|\psi_T(t)U(\pm t)\ u_0\|_{X^{s,1/2}_{mp}} \lesssim \|u_0\|_{H^s}.$$

By Lemma 2.2.2, Proposition 2.2.1, and Corollary 2.2.1,

$$\begin{aligned}
&\left\|\int_0^t U(\pm(t-t'))\, \overline{u(t')}^2 dt'\right\|_{X^{s,1/2}_\mp[0,T]} \\
&\leq \inf_{u'}\left\|\psi_T(t)\int_0^t U(\pm(t-t'))\overline{u'(t')}^2 dt'\right\|_{X^{s,1/2}_\mp} \\
&\lesssim \inf_{u'}\|\overline{u'}^2\|_{X^{s,-1/2}_\mp\cap Y^s_\mp} \\
&\lesssim \inf_{u'} T^\rho\|\overline{u'}\|^2_{X^{s,1/2}_\pm} \\
&= T^\rho\|u\|_{X^{s,1/2}_\mp[0,T]} \\
&\leq T^\rho R^2
\end{aligned}$$

for $0 < \rho < 1/2$. This implies that $\Phi_\pm$ is a map from $B^s_\mp(R,[0,T])$ into itself for some R and T. Moreover, let $u_1, u_2 \in B^s_\mp(R,[0,T])$ and let $u'_1, u'_2 \in X^s_\mp$ satisfy

$$u'_j = u_j \quad \text{on} \quad [0,T]\times\mathbb{R}, \quad \text{supp}\, u'_j \subset [-2T,2T]\times\mathbb{R}.$$

We have, by Lemma 2.2.2,

$$\begin{aligned}
&\|\Phi_\pm(u_1)-\Phi_\pm(u_2)\|_{X_\mp^{s,1/2}[0,T]}\\
&\lesssim \inf_{u_1',u_2'}\left\{\|\overline{(u_1'-u_2')u_1'}\|_{X_\mp^{s,-1/2}\cap Y_\mp^s}+\|\overline{u_2'(u_1'-u_2')}\|_{X_-^{s,-1/2}\cap Y_-^s}\right\}\\
&\le T^\rho \inf_{u_1'-u_2',u_1'}\|\overline{u_1'}\|_{X_\pm^{s,1/2}}\|\overline{u_1'-u_2'}\|_{X_\pm^{s,1/2}}\\
&+T^\rho \inf_{u_2',u_1'-u_2'}\|\overline{u_2'}\|_{X_\pm^{s,1/2}}\|\overline{u_1'-u_2'}\|_{X_\pm^{s,1/2}}\\
&\lesssim T^\rho R\|u_1-u_2\|_{X_\mp^{s,1/2}[0,T]}.
\end{aligned}$$

Therefore, $\Phi_\pm$ is a contraction map on $B_\mp^s(R,[0,T])$ with sufficiently small T and this means we have a unique local solution $u\in X_\mp^{s,1/2}[0,T]$ to (2.3.1) in $H^s(\mathbb{R})$ setting with $s\ge 0$.

2.3.2 Proof of Persistence Regularity

Let $s\ge 0$ and let $u_0\in H^s(\mathbb{R})$. By the proof of Subsection 2.3.1, we have the maximal existence time $T(s')>0$ for $0\le s'\le s$ such that there is a unique local solution $u\in C([0,T(s')),H^{s'}(\mathbb{R}))$. Since $s\ge\lambda(s)$, we have $T(s)\le T(\lambda(s))$, where $\lambda(s)$ is as in (2.2.3). We show that if $T(s)<T(\lambda(s))$, then

$$\sup_{t\in[0,T(s))}\|u(t)\|_{H^s(\mathbb{R})}<\infty, \tag{2.3.2}$$

namely, $T(s)=T(\lambda(s))$ from the point of view of blow-up alternative argument. Let $T_1=\min(1,\frac{T(\lambda(s))-T(s)}{2})$. For sufficiently large C, we define $R_1>0$ as

$$R_1=2C\left(1+\sup_{t\in[0,T(s)+T_1]}\|u(t)\|_{H^{\lambda(s)}(\mathbb{R})}\right)<\infty.$$

We have $0<T_2<T_1$ such that for any $0<T_0<T(s)$ and any $0<T<T_2$, $\Phi_\pm$ is a contraction map on $B_\mp^{\lambda(s)}(R_1,[T_0,T_0+T])$. Let $0<\rho<1/2$, and let $u_1,u_2\in B_\mp^{\lambda(s)}(R_1,[T_0,T_0+T])$. Let $u_1',u_2'\in X_\mp^{s,1/2}$ and $u_1'',u_2''\in X_\mp^{\lambda(s),1/2}$ satisfy

$$\begin{aligned}
&u_j'=u_j \text{ on } [T_0,T_0+T]\times\mathbb{R}, && \operatorname{supp} u_j'\subset[T_0-2T,T_0+2T]\times\mathbb{R},\\
&u_j''=u_j \text{ on } [T_0,T_0+T]\times\mathbb{R}, && \operatorname{supp} u_j''\subset[T_0-2T,T_0+2T]\times\mathbb{R}
\end{aligned}$$

for $j=1,2$. Then, by Proposition 2.2.2,

$$\begin{aligned}
&\|\Phi_\pm(u_1)\|_{X_\mp^{s,1/2}[T_0,\ T_0+T]}\\
&\le\|U(\pm t)u(T_0)\|_{X_\mp^{s,1/2}[T_0,\ T_0+T]}\\
&+\left\|\lambda\int_{T_0}^t U(\pm(t-t'))\overline{u_1(t')}^2dt'\right\|_{X_\mp^{s,1/2}[T_0,\ T_0+T]}\\
&\le C\|u(T_0)\|_{H^s(\mathbb{R})}+CT^\rho\inf_{u_1',u_1''}\|u_1''\|_{X_\mp^{\lambda(s),1/2}}\|u_1'\|_{X_\mp^{s,1/2}}\\
&\le C\|u(T_0)\|_{H^s(\mathbb{R})}+CT^\rho R_1\|u_1\|_{X_\mp^{s,1/2}[T_0,T_0+T]}.
\end{aligned}$$

Let

$$R_2(T_0)=2C(1+\|u(T_0)\|_{H^s(\mathbb{R})})$$

and let

$$T_3 = \min(T_2, (8CR_1)^{-1/\rho}, T(s) - T_0).$$

Then, for $0 < T < T_3$, Φ is a map on

$$B_{\mp}^{\lambda(s)}(R_1, [T_0, T_0 + T]) \cap B_{\mp}^{s}(R_2(T_0), [T_0, T_0 + T]).$$

In addition,

$$\begin{aligned}
&\|\Phi_{\pm}(u_1) - \Phi_{\pm}(u_2)\|_{X_{\mp}^{s,1/2}[T_0,\ T_0+T]} \\
&\leq \left\| \lambda \int_{T_0}^{t} U(\pm(t-t'))\overline{(u_1''(t') + u_2''(t'))(u_1'(t') - u_2'(t'))} \right\|_{X_{\mp}^{s,1/2}[T_0,\ T_0+T]} \\
&\leq CT^{\rho} \inf_{u_1'', u_1' - u_2'} \|u_1''\|_{X_{\mp}^{\lambda(s),1/2}} \|u_1' - u_2'\|_{X_{\mp}^{s,1/2}} \\
&+ CT^{\rho} \inf_{u_2'', u_1' - u_2'} \|u_2''\|_{X_{\mp}^{\lambda(s),1/2}} \|u_1' - u_2'\|_{X_{\mp}^{s,1/2}} \\
&\leq \frac{1}{4}\|u_1 - u_2\|_{X_{\mp}^{s,1/2}[T_0,\ T_0+T]}.
\end{aligned}$$

Therefore, $\Phi_{\pm}$ is a contraction map and u is guaranteed in both $X_{\mp}^{\lambda(s),1/2}[T_0, T_0+T]$ and $X_{\mp}^{s,1/2}[T_0, T_0+T]$. If $T(s) - T_0 < \min(T_2, (8CR_1)^{-1/\rho})$, then $T_3 = T(s) - T_0$ and

$$\sup_{T \in [0, T(s) - T_0)} \|u\|_{X_{\mp}^{s,1/2}[T_0,\ T_0+T]} \leq R_2(T_0),$$

which together with Proposition 2.2.2 implies

$$\sup_{T \in [0, T(s) - T_0)} \|\overline{u}^2\|_{Y_{\mp}^{s}[T_0,\ T_0+T]} \leq CR_2(T_0)^2.$$

Then, by lemma 2.2.5,

$$\sup_{t \in [T_0, T(s))} \|u(t)\|_{H^s(\mathbb{R})} \leq C^2 R_2(T_0)^2.$$

Thus, we obtain (2.3.2) and $T(s) = T(\lambda(s)) = T(0)$.

2.3.3 Proof of Local Well-Posedness without Y Norm

In this subsection, we clarify why the auxiliary space Y is important in our argument. We give an alternative proof of the existence of solutions for $s > 0$, without using the auxiliary norm Y. On the other hand, we shall explain why we need the norm Y at least in our argument if $s = 0$. It is important that $\delta(s)$ in the proof below is strictly positive. We exchange it into the positive power of T. Then, the contraction argument is completed when T is sufficiently small.

The following estimate is used to give a simple proof of Theorem 2.1.1 with $s > 0$.

Proposition 2.3.1. *Let $\varepsilon > 0$, $\rho \geq 0$, $b, \delta \in \mathbb{R}$ satisfy*

$$\begin{aligned}
&1 + b - \delta > \frac{1}{2} + \varepsilon + \rho, \\
&b + \delta + \varepsilon,\ \rho + \delta + \varepsilon \leq 1, \\
&b - \varepsilon,\ b - \rho \geq 0, \\
&s + \varepsilon \geq 1/2.
\end{aligned}$$

Then,

$$\|uv\|_{X^{s,b-1+\delta}_{\pm}} \lesssim \|u\|_{X^{s,b}_{\mp}}\|v\|_{X^{s,b-\rho}_{\mp}} + \|u\|_{X^{s,b-\rho}_{\mp}}\|v\|_{X^{s,b}_{\mp}} \tag{2.3.3}$$

for any $u, v \in X^{s,b}_{\mp}$.

proof. We use the same notation as in the proof of Proposition 2.2.2. We show only

$$\|uv\|_{X^{s,b-1+\delta}_{+}} \lesssim \|u\|_{X^{s,b}_{-}}\|v\|_{X^{s,b-\rho}_{-}} + \|u\|_{X^{s,b-\rho}_{-}}\|v\|_{X^{s,b}_{-}}$$

Since $|\xi|, |\xi-\eta|, |\eta| \leq 3M(\tau,\xi,\sigma,\eta)$ and Lemma 2.2.7,

$$\begin{aligned}
&\left\|\langle\xi\rangle^s \iint_{\mathbb{R}^2} \langle\tau+|\xi|\rangle^{b-1+\delta}\chi_{A_1}(\tau,\xi,\eta,\sigma)\ \widetilde{u}(\tau-\sigma,\xi-\eta)\ \widetilde{v}(\sigma,\eta)\ d\sigma d\eta\right\|_{L^2(\mathbb{R}_\tau\times\mathbb{R}_\xi)} \\
&\lesssim \left\|\int \langle\cdot\rangle^{s-\varepsilon/3}\langle\cdot-\eta\rangle^{-\varepsilon/3}\langle\eta\rangle^{-\varepsilon/3}K_1(\xi,\eta)\ d\eta\right\|_{L^2(\mathbb{R})},
\end{aligned}$$

where

$$K_1(\xi,\eta) = \left\|\langle\cdot+|\xi|\rangle^{b-1+\delta+\varepsilon}\int_{\mathbb{R}} |\widetilde{u}(\cdot-\sigma,\xi-\eta)\widetilde{v}(\sigma,\eta)|\ d\sigma\right\|_{L^2(\mathbb{R})}.$$

By Lemma 2.2.7,

$$K_1(\xi,\eta) \lesssim \|\langle\cdot-|\xi-\eta|\rangle^{b-\rho}\widetilde{u}(\cdot,\xi-\eta)\|_{L^2(\mathbb{R})}\|\langle\cdot-|\eta|\rangle^{b}\widetilde{v}(\cdot,\eta)\|_{L^2(\mathbb{R})}.$$

Similarly, for $j = 2, 3$,

$$\begin{aligned}
&\left\|\langle\xi\rangle^s \iint_{\mathbb{R}^2} \langle\tau+|\xi|\rangle^{b-1+\delta}\chi_{A_j}(\tau,\xi,\eta,\sigma)\ \widetilde{u}(\tau-\sigma,\xi-\eta)\ \widetilde{v}(\sigma,\eta)\ d\sigma d\eta\right\|_{L^2(\mathbb{R}_\tau\times\mathbb{R}_\xi)} \\
&\lesssim \|\int \langle\cdot\rangle^{s-\varepsilon/3}\langle\cdot-\eta\rangle^{-\varepsilon/3}\langle\eta\rangle^{-\varepsilon/3}K_j(\xi,\eta)\ d\eta\|_{L^2(\mathbb{R})},
\end{aligned}$$

where

$$\begin{aligned}
K_2 &= \left\|\langle\cdot+|\xi|\rangle^{b-1+\delta}\int_{\mathbb{R}}\langle\cdot-|\xi-\eta|\rangle^{\varepsilon}|\widetilde{u}(\cdot-\sigma,\xi-\eta)\ \widetilde{v}(\sigma,\eta)|\ d\sigma\right\|_{L^2(\mathbb{R})}, \\
K_3 &= \left\|\langle\cdot+|\xi|\rangle^{b-1+\delta}\int_{\mathbb{R}}|\widetilde{u}(\cdot-\sigma,\xi-\eta)\langle\cdot-|\eta|\rangle^{\varepsilon}\widetilde{v}(\sigma,\eta)|\ d\sigma\right\|_{L^2(\mathbb{R})}.
\end{aligned}$$

By Lemma 2.2.7,

$$\begin{aligned}
K_2(\xi,\eta) &\lesssim \|\langle\cdot-|\xi-\eta|\rangle^{b}\widetilde{u}(\cdot,\xi-\eta)\|_{L^2(\mathbb{R})}\|\langle\cdot-|\eta|\rangle^{b-\rho}\widetilde{v}(\cdot,\eta)\|_{L^2(\mathbb{R})}, \\
K_3(\xi,\eta) &\lesssim \|\langle\cdot-|\xi-\eta|\rangle^{b-\rho}\widetilde{u}(\cdot,\xi-\eta)\|_{L^2(\mathbb{R})}\|\langle\cdot-|\eta|\rangle^{b}\widetilde{v}(\cdot,\eta)\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Then, we obtain (2.3.3) by Lemma 1.3.11. Q.E.D.

Remark 2.3.2. $b = 1/2$, $\delta = 0$, $\varepsilon = 1/2$ *are the only numbers that ensures* (2.3.3) *for* $s = 0$. *See Proposition 2.3.3.*

A simple proof of existence of solution for $s > 0$. Let $s > 0$, $u_0 \in H^s$ and let $0 < T \le 1$. We take $b(s) = \min(3/4, (1+s)/2) > 1/2$ and $\delta(s) = \min(1/4, s/2) > 0$ for Proposition 2.3.1.

We define a metric space

$$B'^s_\mp(R,T) = \{u \in X'^{s,b(s)}_\mp[0,T] \ ; \ \|u\|_{X'^{s,b(s)}_\mp[0,T]} \le R\}$$

with metric

$$d'_\mp(u_1,u_2) = \|u_1 - u_2\|_{X'^{s,b(s)}_\mp[0,T]}.$$

We see $(B'^s_\mp(R,T), d'_\mp)$ is a complete metric space. We prove that $\Phi_\pm$ defined as (2.3.1) is a contraction map on $B'^s_\mp(R,T)$ for sufficiently large R and sufficiently small T.

Let $u \in B'^s_\mp(R,T)$ and let $u' \in X^{s,b(s)}_\mp$ satisfy

$$u' = u \quad \text{on} \quad [0,T]\times\mathbb{R}.$$

We have

$$\begin{aligned}\|\Phi_\pm(u)\|_{X'^{s,b(s)}_\mp[0,T]} &\le \|U(\pm t)\ u_0\|_{X'^{s,b(s)}_\mp[0,T]} \\ &\quad + \left\|\lambda\int_0^t U(\pm(t-t'))\ \overline{u(t')}^2 dt'\right\|_{X'^{s,b(s)}_\mp[0,T]}.\end{aligned}$$

By Lemma 2.2.4,

$$\|U(\pm t)\ u_0\|_{X'^{s,b}_\mp[0,T]} \le \|\psi(t)U(\pm t)\ u_0\|_{X'^{s,b}_\mp} \lesssim \|u_0\|_{H^s(\mathbb{R})}.$$

By Propositions 2.2.1 and 2.3.1, we obtain

$$\begin{aligned}&\left\|\int_0^t U(\pm(t-t'))\overline{u(t')}^2 dt'\right\|_{X'^{s,b(s)}_\mp[0,T]} \\ &\le \inf_{u'}\left\|\psi_T(t)\int_0^t U(\pm(t-t'))\overline{u'(t')}^2\ dt'\right\|_{X^{s,b(s)}_\mp} \\ &\lesssim \inf_{u'} T^{\delta(s)}\|\overline{u'}^2\|_{X^{s,b(s)-1+\delta(s)}_\mp} \\ &\lesssim \inf_{u'} T^{\delta(s)}\|u'\|^2_{X^{s,b(s)}_\mp} \\ &\lesssim T^{\delta(s)}\|u\|_{X^{s,b(s)}_\mp[0,T]} \\ &\le T^{\delta(s)}R^2.\end{aligned}$$

Thus, $\Phi_\pm$ is a map from $B'^s_\mp(R,T)$ to $B'^s_\mp(R,T)$ for some R and T. Moreover, let $u_1, u_2 \in B'^s_\mp(R,T)$ and let $u'_1, u'_2 \in X^{s,b(s)}_\mp$ satisfy

$$u'_j = u_j \quad \text{on} \quad [0,T]\times\mathbb{R}$$

for $j = 1,2$. Then, we have

$$\begin{aligned}&\|\Phi_\mp(u_1) - \Phi_\mp(u_2)\|_{X'^{s,b(s)}_\mp[0,T]} \\ &\lesssim \inf_{u'_1,u'_2} T^{\delta(s)}\|\overline{(u'_1+u'_2)(u'_1-u'_2)}\|_{X^{s,b(s)-1+\delta(s)}_-} \\ &\lesssim T^{\delta(s)} \inf_{u'_1,u'_1-u'_2}\|u'_1\|_{X^{s,b(s)}_\mp}\|u'_1-u'_2\|_{X^{s,b(s)}_\mp} \\ &+ T^{\delta(s)} \inf_{u'_2,u'_1-u'_2}\|u'_2\|_{X^{s,b(s)}_\mp}\|u'_1-u'_2\|_{X^{s,b(s)}_\mp} \\ &\lesssim T^{\delta(s)}R\|u_1-u_2\|_{X'^{s,b(s)}_\mp[0,T]}.\end{aligned}$$

Thus, $\Phi_\pm$ is a contraction map on $B'^s_\mp(R,T)$ for sufficiently small T. Q.E.D.

The following proposition implies that we can not take $\delta > 0$ when $s = 0$ in the above proof.

Proposition 2.3.3. *For any* $b \in [0,1/2) \cup (1/2,1]$*, there exists a pair* $(u,v) \in X_-^{0,b} \times X_-^{0,b}$ *such that*

$$\|uv\|_{X_+^{0,b-1}} = \infty. \tag{2.3.4}$$

Also for any $\delta > 0$*, there exists a pair* $(u,v) \in X_-^{0,1/2} \times X_-^{0,1/2}$ *such that*

$$\|uv\|_{X_+^{0,-1/2+\delta}} = \infty. \tag{2.3.5}$$

Remark 2.3.4. *This is the reason why we use not only the norm* $X_\pm^{s,b}$ *but also the norm* $Y_\pm^s$*, since Proposition 2.2.1 requires* $Y_\pm^s$ *norm when* $b = 1/2$*. Moreover, support restricted functions is necessary to estimate Duhamel term by time in order to apply contraction argument to Cauchy problem* (2.1.1)*, since Proposition 2.2.1 doesn't give such an estimate when* $b = 1/2$*.*

Proof of Proposition 2.3.3. Suppose $1/2 < b \le 1$. Let $0 < 2\varepsilon \le b - 1/2$ and let

$$\widetilde{u_1}(\tau,\xi) = \widetilde{v_1}(\tau,\xi) = \langle \xi \rangle^{-\frac{1}{2}-\varepsilon} \langle \tau - |\xi| \rangle^{-b-1/2-\varepsilon}.$$

If $\tau > 2$, $\tau - 1 < \xi < \tau + 1$, then

$$\begin{aligned}
&\langle \tau + |\xi| \rangle^{b-1} \iint_{\mathbb{R}^2} \langle \eta \rangle^{-\frac{1}{2}-\varepsilon} \langle \xi - \eta \rangle^{-1/2-\varepsilon} \langle \sigma - |\eta| \rangle^{-b-1/2-\varepsilon} \langle \tau - \sigma - |\xi - \eta| \rangle^{-b-1/2-\varepsilon} d\sigma d\eta \\
&\gtrsim \langle 2\tau + 1 \rangle^{b-1} \int_0^\xi \langle \eta \rangle^{-\frac{1}{2}-\varepsilon} \langle \xi - \eta \rangle^{-\frac{1}{2}-\varepsilon} \langle \tau - |\xi - \eta| - |\eta| \rangle^{-2b-2\varepsilon} d\eta \\
&\gtrsim \langle 2\tau + 1 \rangle^{b-1} \int_0^\xi (1 + \xi + \eta(\xi - \eta))^{-1/2-\varepsilon} d\eta \\
&\gtrsim \langle 2\tau + 1 \rangle^{b-1} \int_0^\xi \langle \xi \rangle^{-1-2\varepsilon} d\eta \\
&\gtrsim \langle \tau + 1 \rangle^{-1/2}.
\end{aligned}$$

This implies $u_1 v_1 \notin X_+^{0,b-1}$. Moreover, suppose $0 \le b < 1/2$. Let b and δ satisfy $0 < 2\varepsilon \le 1/2 - b$ and let

$$\begin{aligned}
\widetilde{u_2}(\tau,\xi) &= \langle \xi \rangle^{-\frac{1}{2}-\varepsilon} \langle \tau - |\xi| \rangle^{-b-1/2-\varepsilon}, \\
\widetilde{v_2}(\tau,\xi) &= \langle \xi \rangle^{-\frac{1}{2}-\varepsilon} \langle \tau - |\xi| \rangle^{-b} \langle \tau + |\xi| \rangle^{-1/2-\varepsilon}.
\end{aligned}$$

Since for any real number a and b, $\langle a + b \rangle \le \langle a \rangle \langle b \rangle$, for $\xi > 0$,

$$\begin{aligned}
&\langle \tau + |\xi| \rangle^{b-1} \iint_{\mathbb{R}^2} \langle \eta \rangle^{-\frac{1}{2}-\varepsilon} \langle \xi - \eta \rangle^{-1/2-\varepsilon} \langle \sigma - |\eta| \rangle^{-b-1/2-\varepsilon} \\
&\qquad \cdot \langle \tau - \sigma - |\xi - \eta| \rangle^{-b} \langle \tau - \sigma + |\xi - \eta| \rangle^{-1/2-\varepsilon} d\sigma d\eta \\
&\gtrsim \langle \tau + |\xi| \rangle^{b-1} \iint_{\mathbb{R}^2} \langle \eta \rangle^{-\frac{1}{2}-\varepsilon} \langle \xi - \eta \rangle^{-b-1/2-\varepsilon} \langle \sigma - |\eta| \rangle^{-b-1/2-\varepsilon} \\
&\qquad \cdot \langle \tau - \sigma + |\xi - \eta| \rangle^{-b-1/2-\varepsilon} d\sigma d\eta \\
&\gtrsim \langle \tau + \xi \rangle^{b-1} \int_{-\infty}^0 \langle \eta \rangle^{-\frac{1}{2}-\varepsilon} \langle \xi - \eta \rangle^{-b-\frac{1}{2}-\varepsilon} \langle \tau + \xi \rangle^{-2b-2\varepsilon} d\eta \\
&\gtrsim \langle \tau + \xi \rangle^{-b-1-2\varepsilon} \langle \xi \rangle^{-1/2} \notin L^2_{\xi>0}(L^2_\tau).
\end{aligned}$$

Therefore, $u_2v_2 \notin X_+^{0,b-1}$. We complete the proof of (2.3.4).

Suppose $\delta > 0$ and $b = 1/2$. Let ε satisfy $0 < 2\varepsilon \le \delta$ and let

$$\widetilde{u_3}(\tau,\xi) = \widetilde{v_3}(\tau,\xi) = \langle\xi\rangle^{-\frac{1}{2}-\varepsilon}\langle\tau - |\xi|\rangle^{-1-\varepsilon}.$$

If $\tau > 2$, $\tau - 1 < \xi < \tau + 1$, then

$$\begin{aligned}
&\langle\tau + |\xi|\rangle^{-1/2+\delta} \iint_{\mathbb{R}^2} \langle\eta\rangle^{-\frac{1}{2}-\varepsilon}\langle\xi-\eta\rangle^{-1/2-\varepsilon}\langle\sigma - |\eta|\rangle^{-1-\varepsilon}\langle\tau-\sigma-|\xi-\eta|\rangle^{-1-\varepsilon}d\sigma d\eta \\
&\gtrsim \langle 2\tau+1\rangle^{-1/2+\delta}\int_0^\xi \langle\eta\rangle^{-\frac{1}{2}-\varepsilon}\langle\xi-\eta\rangle^{-\frac{1}{2}-\varepsilon}\langle\tau-|\xi-\eta|-|\eta|\rangle^{-1-2\varepsilon}d\eta \\
&\gtrsim \langle 2\tau+1\rangle^{-1/2+\delta}\int_0^\xi (1+\xi+\eta(\xi-\eta))^{-1/2-\varepsilon}d\eta \\
&\gtrsim \langle 2\tau+1\rangle^{-1/2+\delta}\int_0^\xi \langle\xi\rangle^{-1-2\varepsilon}d\eta \\
&\gtrsim \langle\tau+1\rangle^{-1/2}.
\end{aligned}$$

This yields $u_3v_3 \notin X_+^{0,b-1}$ and we obtain (2.3.5). Q.E.D.

Corollary 2.3.1. *For any $b \in \mathbb{R}$ and $s < 0$, there exists a pair $u, v \in X_-^{s,b}$ such that*

$$\|uv\|_{X_+^{s,b-1}} = \infty. \tag{2.3.6}$$

Remark 2.3.5. *Proposition 2.3.3 and Corollary 2.3.1 show that Proposition 2.2.2 is almost optimal.*

Proof of Corollary 2.3.1. Suppose $1/2 \le b \le 1$. Let $0 < \varepsilon < -s$ and let

$$\widetilde{u_1}(\tau,\xi) = \widetilde{v_1}(\tau,\xi) = \langle\xi\rangle^{-\frac{1}{2}-\varepsilon}\langle\tau-|\xi|\rangle^{-b-1/2-\varepsilon}.$$

If $\tau > 2$, $\tau - 1 < \xi < \tau + 1$, then

$$\begin{aligned}
&\langle\xi\rangle^s\langle\tau+|\xi|\rangle^{b-1} \\
&\cdot\iint_{\mathbb{R}^2}\langle\eta\rangle^{-s-\frac{1}{2}-\varepsilon}\langle\xi-\eta\rangle^{-s-1/2-\varepsilon}\langle\sigma-|\eta|\rangle^{-b-1/2-\varepsilon}\langle\tau-\sigma-|\xi-\eta|\rangle^{-b-1/2-\varepsilon}d\sigma d\eta \\
&\gtrsim \langle\xi\rangle^s\langle 2\tau+1\rangle^{b-1}\int_0^\xi\langle\eta\rangle^{-s-\frac{1}{2}-\varepsilon}\langle\xi-\eta\rangle^{-s-\frac{1}{2}-\varepsilon}\langle\tau-|\xi-\eta|-|\eta|\rangle^{-2b-2\varepsilon}d\eta \\
&\gtrsim \langle\xi\rangle^s\langle 2\tau+1\rangle^{b-1}\int_0^\xi(1+\xi+\eta(\xi-\eta))^{-s-1/2-\varepsilon}d\eta \\
&\gtrsim \langle 2\tau+1\rangle^{b-1}\int_0^\xi\langle\xi\rangle^{-2s-1-2\varepsilon}d\eta \\
&\gtrsim \langle\tau+1\rangle^{-1/2}.
\end{aligned}$$

This implies $u_1v_1 \notin X_+^{s,b-1}$. Moreover, suppose $0 \le b < 1/2$. Let b and δ satisfy $0 < 2\varepsilon \le 1/2 - b$ and let

$$\begin{aligned}
\widetilde{u_2}(\tau,\xi) &= \langle\xi\rangle^{-s-\frac{1}{2}-\varepsilon}\langle\tau-|\xi|\rangle^{-b-1/2-\varepsilon}, \\
\widetilde{v_2}(\tau,\xi) &= \langle\xi\rangle^{-s-\frac{1}{2}-\varepsilon}\langle\tau-|\xi|\rangle^{-b}\langle\tau+|\xi|\rangle^{-1/2-\varepsilon}.
\end{aligned}$$

Since for any real number a and b, $\langle a+b\rangle \le \langle a\rangle\langle b\rangle$, for $\xi>0$,

$$\begin{aligned}
&\langle\xi\rangle^s\langle\tau+|\xi|\rangle^{b-1}\iint_{\mathbb{R}^2}\langle\eta\rangle^{-s-\frac12-\varepsilon}\langle\xi-\eta\rangle^{-s-1/2-\varepsilon}\langle\sigma-|\eta|\rangle^{-b-1/2-\varepsilon}\\
&\qquad\qquad\cdot\langle\tau-\sigma-|\xi-\eta|\rangle^{-b}\langle\tau-\sigma+|\xi-\eta|\rangle^{-1/2-\varepsilon}d\sigma d\eta\\
&\gtrsim\langle\tau+|\xi|\rangle^{b-1}\iint_{\mathbb{R}^2}\langle\eta\rangle^{-\frac12-\varepsilon}\langle\xi-\eta\rangle^{-b-1/2-\varepsilon}\langle\sigma-|\eta|\rangle^{-b-1/2-\varepsilon}\\
&\qquad\qquad\cdot\langle\tau-\sigma+|\xi-\eta|\rangle^{-b-1/2-\varepsilon}d\sigma d\eta\\
&\gtrsim\langle\tau+\xi\rangle^{b-1}\int_{-\infty}^0\langle\eta\rangle^{-\frac12-\varepsilon}\langle\xi-\eta\rangle^{-b-\frac12-\varepsilon}\langle\tau+\xi\rangle^{-2b-2\varepsilon}d\eta\\
&\gtrsim\langle\tau+\xi\rangle^{-b-1-2\varepsilon}\langle\xi\rangle^{-1/2}\notin L^2_{\xi>0}(L^2_\tau).
\end{aligned}$$

Therefore, $u_2v_2\notin X_+^{0,b-1}$. We complete the proof of (2.3.6). Q.E.D.

2.4 Proof of Theorem 2.1.2

In this section, we prove Theorem 2.1.2 by direct calculation. For simplicity, we show only (2.1.4) with $U(t)$. Let $-1/2<s<0$. Let $u_0=\mathfrak{F}^{-1}\chi_{\xi>1}\langle\xi\rangle^{-1/2-s-\varepsilon}$.

$$\begin{aligned}
&\mathfrak{F}\Big[\int_0^t U(t-t')\Big(\overline{U(t')u_0}\int_0^{t'}U(t''-t')(U(t')u_0)^2dt''\Big)\,dt'\Big](\xi)\\
&=\exp(it\sqrt{m^2+\xi^2})\int_0^t\int_{\mathbb{R}}\exp(-it'g(\xi,\eta_1))\overline{\hat{u}_0(-\xi+\eta_1)}\hat{J}(t',\eta_1)d\eta_1dt',
\end{aligned}$$

where

$$g(\xi,\eta_1)=\sqrt{m^2+\xi^2}+\sqrt{m^2+(\xi-\eta_1)^2}+\sqrt{m^2+\eta_1^2} \tag{2.4.1}$$

and

$$\hat{J}(t,\eta_1)=\int_0^t\int_{\mathbb{R}}\exp(it'g(\eta_1,\eta_2))\hat{u}_0(\eta_1-\eta_2)\hat{u}_0(\eta_2)d\eta_2dt'.$$

Let $(u_{0,n})$ be a sequence of $\mathscr{S}(\mathbb{R})$ such that $\hat{u}_{0,n}$ is non-negative function and $\hat{\phi}_n(\xi)$ converges $\hat{u}_0(\xi)$ from below monotonically for any ξ. Let

$$\begin{aligned}
J_n(t,\eta_1)&=-i\big(J_{n,1}(t,\eta_1)-J_{n,2}(t,\eta_1)\big),\\
\hat{J}_{n,1}(t,\eta_1)&=\int_{\mathbb{R}}\frac{1}{g(\eta_1,\eta_2)}\hat{u}_{0,n}(\eta_1-\eta_2)\hat{u}_{0,n}(\eta_2)d\eta_2,\\
\hat{J}_{n,2}(t,\eta_1)&=\int_{\mathbb{R}}\frac{\exp(itg(\eta_1,\eta_2))}{g(\eta_1,\eta_2)}\hat{u}_{0,n}(\eta_1-\eta_2)\hat{u}_{0,n}(\eta_2)d\eta_2.
\end{aligned}$$

Then,

$$\Big\|\chi_{|\cdot|\le1}\int_{-1}^1\frac{\exp(itg(\cdot,\eta_2))}{g(\cdot,\eta_2)}\hat{u}_{0,n}(\cdot-\eta_2)\hat{u}_{0,n}(\eta_2)d\eta_2\Big\|_{L^2(\mathbb{R})}\lesssim\|u_{0,n}\|^2_{H^s(\mathbb{R})}$$

and

$$\begin{aligned}
&\left\| \chi_{|\cdot|\ge 1} \int_{|\eta_2|\ge 1} \frac{\exp(itg(\cdot,\eta_2))}{g(\cdot,\eta_2)} \hat{u}_{0,n}(\cdot-\eta_2)\hat{u}_{0,n}(\eta_2)d\eta_2 \right\|_{L^2(\mathbb{R})} \\
&\lesssim \left\| \langle\cdot\rangle^s \int_{\mathbb{R}} \frac{1}{1+g(\cdot,\eta_2)} \hat{u}_{0,n}(\xi-\eta_2)\hat{u}_{0,n}(\eta_2)d\eta_2 \right\|_{L^2(\mathbb{R})} \\
&\lesssim \left\| \langle\cdot\rangle^s \int \langle\cdot-\eta_2\rangle^{-1/2}\langle\eta_2\rangle^{-1/2} \hat{u}_{0,n}(\cdot-\eta_2)\hat{u}_{0,n}(\eta_2)d\eta_2 \right\|_{L^2(\mathbb{R})} \\
&\lesssim \|u_{0,n}\|^2_{H^s(\mathbb{R})}.
\end{aligned}$$

Therefore $\|J_2(t,\cdot)\|_{H^s(\mathbb{R})} \lesssim \|u_{0,n}\|^2_{H^s(\mathbb{R})}$. Similarly,

$$\begin{aligned}
&\left\| \langle\cdot\rangle^s \int_0^t \int_{\mathbb{R}} \exp(-it'g(\cdot,\eta_1))\overline{\hat{u}_{0,n}(-\cdot+\eta_1)}\hat{J}_{n,1}(t',\eta_1)d\eta_1 dt' \right\|_{L^2(\mathbb{R})} \\
&\le \left\| \langle\cdot\rangle^s \int_{\mathbb{R}} \frac{2}{g(\cdot,\eta_1)}\overline{\hat{u}_{0,n}(-\cdot+\eta_1)}\hat{J}_{n,1}(t',\eta_1)d\eta_1 dt' \right\|_{L^2(\mathbb{R})} \\
&\lesssim \|u_{0,n}\|_{H^s(\mathbb{R})}\|J_{n,1}\|_{H^s(\mathbb{R})} \\
&\lesssim \|u_{0,n}\|^3_{H^s(\mathbb{R})}.
\end{aligned}$$

On the other hand, let

$$\begin{aligned}
&b(\xi,\eta_1,\eta_2) \\
&= g(\xi,\eta_1) - g(\eta_1,\eta_2) \\
&= \sqrt{m^2+\xi^2} + \sqrt{m^2+(\xi-\eta_1)^2} - \sqrt{m^2+(\eta_1-\eta_2)^2} - \sqrt{m^2+\eta_2^2}.
\end{aligned}$$

Then, $|b(\xi,\eta_1,\eta_2)| < 2m$ when $\xi > 0$, $\xi-\eta_1 < 0$, $\eta_1-\eta_2 < 0$, and $\eta_2 < 0$. Indeed, if $b(\xi,\eta_1,\eta_2) > 0$, then

$$\begin{aligned}
&|b(\xi,\eta_1,\eta_2)| \\
&= \sqrt{m^2+\xi^2} + \sqrt{m^2+(\xi-\eta_1)^2} - \sqrt{m^2+(\eta_1-\eta_2)^2} - \sqrt{m^2+\eta_2^2} \\
&\le (m+|\xi|) + (m+|\xi-\eta_1|) - (|\eta_1-\eta_2|) - (|\eta_2|) \\
&= 2m,
\end{aligned}$$

and if $b(\xi,\eta_1,\eta_2) < 0$, then

$$\begin{aligned}
&|b(\xi,\eta_1,\eta_2)| \\
&= -\sqrt{m^2+\xi^2} - \sqrt{m^2+(\xi-\eta_1)^2} + \sqrt{m^2+(\eta_1-\eta_2)^2} + \sqrt{m^2+\eta_2^2} \\
&\le -\xi| - |\xi-\eta_1| + (m+|\eta_1-\eta_2|) + (m+|\eta_2|) \\
&= 2m.
\end{aligned}$$

If $\xi > 1$ and t is sufficiently small, then $\operatorname{Re}\exp(itb(\xi,\eta_1,\eta_2)) > 1/2$ and

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\left\|\langle\cdot\rangle^s\operatorname{Re}\int_0^t\int_{\mathbb{R}}\exp(-it'g(\cdot,\eta_1))\overline{\hat{u}_{0,n}(-\cdot+\eta_1)}\hat{J}_{n,2}(t',\eta_1)d\eta_1dt'\right\|_{L^2(\mathbb{R})}\\
&=\sup_{n\in\mathbb{N}}\left\|\langle\cdot\rangle^s\operatorname{Re}\int_0^t\int_{\mathbb{R}}\exp(-it'b(\cdot,\eta_1,\eta_2))\overline{\hat{u}_{0,n}(-\cdot+\eta_1)}\int\frac{1}{g(\eta_1,\eta_2)}\hat{u}_{0,n}(\eta_1-\eta_2)\hat{u}_{0,n}(\eta_2)d\eta_2d\eta_1dt'\right\|_{L^2(\mathbb{R})}\\
&\gtrsim\sup_{n\in\mathbb{N}}\left\|\langle\cdot\rangle^s\int_{\mathbb{R}}\hat{u}_{0,n}(-\cdot+\eta_1)\langle\eta_1\rangle^{-1}\int_0^{\eta_1}\hat{u}_{0,n}(\eta_1-\eta_2)\hat{u}_{0,n}(\eta_2)d\eta_2d\eta_1\right\|_{L^2(\mathbb{R})}\\
&\gtrsim\left\|\langle\cdot\rangle^s\int_\cdot^\infty\langle\cdot-\eta_1\rangle^{-s-1/2-\varepsilon}\langle\eta_1\rangle^{-1}\int_0^{\eta_1}\langle\eta_1-\eta_2\rangle^{-s-1/2-\varepsilon}\langle\eta_2\rangle^{-s-1/2-\varepsilon}\ d\eta_2d\eta_1\right\|_{L^2(\mathbb{R})}\\
&\gtrsim\left\|\langle\cdot\rangle^s\int_\cdot^\infty\langle\cdot-\eta_1\rangle^{-s-1/2-\varepsilon}\langle\eta_1\rangle^{-2s-1-2\varepsilon}d\eta_1\right\|_{L^2(\mathbb{R})}\\
&\gtrsim\left\|\langle\cdot\rangle^s\int_\cdot^\infty\langle\eta_1\rangle^{-3s-3/2-3\varepsilon}d\eta_1\right\|_{L^2(\mathbb{R})}\\
&\gtrsim\|\langle\cdot\rangle^{-2s-1/2-3\varepsilon}\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Since $\varepsilon\le -2s/3$, $-2s-1/2-3\varepsilon\ge -1/2$,

$$\int_0^t U(t-t')\left(\overline{U(t')u_0}\int_0^{t'}U(t''-t')(U(t')u_0)^2dt''\right)\,dt'\notin H^s(\mathbb{R}).$$

2.5 Proof of Theorem 2.1.3

In this section, we prove Theorem 2.1.3 by direct calculation. At first, we decompose the Duhamel term with linear solution into 4 terms as follows:

$$\mathfrak{F}\left[\int_0^t U(\pm(t-t'))\overline{U(\pm t')u_{0,k}}^2dt'\right](\xi)=\sum_{j=1}^4\hat{J}_j(t,\xi),$$

where

$$\begin{aligned}
\hat{J}_1(t,\xi)&=k^{-2s}\int_{\mathbb{R}}\frac{\exp(\pm itg(\xi,\eta))-1}{ig(\xi,\eta)}\chi_{[-1,1]}(\xi-\eta+k)\chi_{[-1,1]}(\eta+k)d\eta dt',\\
\hat{J}_2(t,\xi)&=k^{-2s}\int_{\mathbb{R}}\frac{\exp(\pm itg(\xi,\eta))-1}{ig(\xi,\eta)}\chi_{[-1,1]}(\xi-\eta+k)\chi_{[-1,1]}(\eta-k)d\eta dt',\\
\hat{J}_3(t,\xi)&=k^{-2s}\int_{\mathbb{R}}\frac{\exp(\pm itg(\xi,\eta))-1}{ig(\xi,\eta)}\chi_{[-1,1]}(\xi-\eta-k)\chi_{[-1,1]}(\eta+k)d\eta dt',\\
\hat{J}_4(t,\xi)&=k^{-2s}\int_{\mathbb{R}}\frac{\exp(\pm itg(\xi,\eta))-1}{ig(\xi,\eta)}\chi_{[-1,1]}(\xi-\eta-k)\chi_{[-1,1]}(\eta-k)d\eta dt'
\end{aligned}$$

and g is defined in (2.4.1). If $\hat{J}_1\neq 0$, then $\xi\sim -2k$, $\eta,\xi-\eta\sim -k$, and therefore $\|J_1(t,\cdot)\|_{H^s(\mathbb{R})}\lesssim k^{-s-1}$. If $\hat{J}_4\neq 0$, then $\xi\sim 2k$, $\eta,\xi-\eta\sim k$, and therefore $\|J_4(t,\cdot)\|_{H^s(\mathbb{R})}\lesssim k^{-s-1}$. Moreover, $J_2=J_3$ and if $\hat{J}_2\neq 0$, then $\xi-\eta\sim -k$ and $\eta\sim k$, and therefore $k\sim 1$ and $\|J_2(t,\cdot)\|_{H^s(\mathbb{R})}\gtrsim n^{-2s-1}$. This means

$$\left\|\int_0^t U(\pm(t-t'))\overline{U(\pm t')u_{0,k}}^2dt'\right\|\gtrsim k^{-2s-1}\to\infty$$

as $k\to\infty$.

2.6 Proof of Theorem 2.1.4

In this section, we prove Theorem 2.1.4 by direct calculation.

2.6.1 (2.1.2) with $-1/2 < s < 1/2$

We estimate

$$\begin{aligned}
&\mathfrak{F}\Big[\int_0^t U(\pm(t-t'))\big(\overline{U(\pm t')u_0}U(\pm t')u_0\big)dt'\Big](\xi)\\
&=\int_0^t\int_{\mathbb{R}}\exp(-it'\big(\sqrt{m^2+\xi^2}+\sqrt{m^2+(\xi-\eta)^2}-\sqrt{m^2+\eta^2}\big))\\
&\qquad\cdot\overline{\hat{u}_0(\eta-\xi)}\hat{u}_0(\eta)\ d\eta dt'.
\end{aligned}$$

For preparation, we remind the estimate

$$-m\le\sqrt{m^2+\xi^2}+\sqrt{m^2+(\eta-\xi)^2}-\sqrt{m^2+\eta^2}\le 2m$$

for $0\le\xi\le\eta$. Indeed,

$$\begin{aligned}
&\sqrt{m^2+\xi^2}+\sqrt{m^2+(\eta-\xi)^2}-\sqrt{m^2+\eta^2}\\
&\ge\xi+\eta-\xi-\sqrt{m^2+\eta^2}\\
&\ge\xi+\eta-\xi-\eta-m=-m
\end{aligned}$$

and

$$\begin{aligned}
&\sqrt{m^2+\xi^2}+\sqrt{m^2+(\eta-\xi)^2}-\sqrt{m^2+\eta^2}\\
&\le 2m+\xi+\eta-\xi-\sqrt{m^2+\eta^2}\\
&\le 2m+\xi+\eta-\xi-\eta=2m.
\end{aligned}$$

Let $\hat{u}_0=\chi_{[0,\infty)}\langle\cdot\rangle^{-s-1/2-\varepsilon}$, where $\varepsilon>0$. We estimate the Duhamel term by duality with $\mathfrak{F}^{-1}[\chi_{[0,\infty)}\langle\cdot\rangle^{s-1/2-\varepsilon}]\in H^{-s}(\mathbb{R})$. In particular, we estimate the following dual product:

$$\begin{aligned}
&\mathrm{Re}\Big\langle\mathfrak{F}^{-1}[\chi_{[0,\infty)}\langle\cdot\rangle^{-s-1/2-\varepsilon}]\Big|\int_0^t U(\pm(t-t'))\big(\overline{U(\pm t')u_0}U(\pm t')u_0\big)dt'\Big\rangle_{H^s(\mathbb{R})}\\
&=\mathrm{Re}\int_0^\infty\int_0^t\int_\xi^\infty\exp(-it'(\sqrt{m^2+\xi^2}+\sqrt{m^2+(\xi-\eta)^2}-\sqrt{m^2+\eta^2}))dt'\\
&\qquad\cdot\langle\xi\rangle^{s-1/2-\varepsilon}\langle\xi-\eta\rangle^{-s-1/2-\varepsilon}\langle\eta\rangle^{-s-1/2-\varepsilon}\ d\eta\ d\xi.
\end{aligned}$$

For sufficiently small t and $0\le\xi\le\eta$,

$$\mathrm{Re}\Big(\exp(-it'(\sqrt{m^2+\xi^2}+\sqrt{m^2+(\xi-\eta)^2}-\sqrt{m^2+\eta^2})\Big)\ge 1/2.$$

Then,

$$\begin{aligned}
&\mathrm{Re}\Big\langle \mathfrak{F}^{-1}[\chi_{[0,\infty)}\langle\cdot\rangle^{-s-1/2-\varepsilon}]\Big| \int_0^t U(\pm(t-t'))\big(\overline{U(\pm t')u_0}U(\pm t')u_0\big)dt'\Big\rangle_{H^s(\mathbb{R})}\\
&\geq t/2\int_0^\infty\int_\xi^\infty \langle\xi\rangle^{s-1/2-\varepsilon}\langle\xi-\eta\rangle^{-s-1/2-\varepsilon}\langle\eta\rangle^{-s-1/2-\varepsilon}\ d\eta\ d\xi\\
&\geq t/2\int_0^\infty\int_\xi^\infty \langle\xi\rangle^{s-1/2-\varepsilon}\langle\eta\rangle^{-2s-1-\varepsilon}\ d\eta\ d\xi\\
&\gtrsim t\int_0^\infty \langle\xi\rangle^{-s-1/2-3\varepsilon} = \infty.
\end{aligned}$$

for $\varepsilon \leq 1/6 - s/3$.

2.6.2 (2.1.3) with $-1/2 < s < 1/2$

We estimate

$$\begin{aligned}
&\mathfrak{F}\Big[\int_0^t U(\pm(t-t'))\big(U(\pm t')u_0\big)^2 dt'\Big](\xi)\\
&=\int_0^t\int_{\mathbb{R}} \exp(-it'\big(\sqrt{m^2+\xi^2}-\sqrt{m^2+(\xi-\eta)^2}-\sqrt{m^2+\eta^2}\big))\hat{u}_0(\eta)^2\ d\eta dt'.
\end{aligned}$$

For preparation, we remind the estimate

$$-2m \leq \sqrt{m^2+\xi^2}-\sqrt{m^2+(\eta-\xi)^2}-\sqrt{m^2+\eta^2} \leq m$$

for $0 \leq \eta \leq \xi$. Indeed,

$$\begin{aligned}
&\sqrt{m^2+\xi^2}-\sqrt{m^2+(\xi-\eta)^2}-\sqrt{m^2+\eta^2}\\
&\geq \xi-\sqrt{m^2+(\xi-\eta)^2}-\sqrt{m^2+\eta^2}\\
&\geq \xi-m-\xi+\eta-\eta-m=-2m
\end{aligned}$$

and

$$\begin{aligned}
&\sqrt{m^2+\xi^2}-\sqrt{m^2+(\xi-\eta)^2}-\sqrt{m^2+\eta^2}\\
&\leq m+\xi+\sqrt{m^2+(\xi-\eta)^2}-\sqrt{m^2+\eta^2}\\
&\leq m+\xi-\xi+\eta-\eta=m.
\end{aligned}$$

Let $\hat{u}_0 = \chi_{[0,\infty)}\langle\cdot\rangle^{-s-1/2-\varepsilon}$, where $\varepsilon > 0$. We estimate the Duhamel term by duality with $\mathfrak{F}^{-1}[\chi_{[0,\infty)}\langle\cdot\rangle^{s-1/2-\varepsilon}] \in H^{-s}(\mathbb{R})$ and a restriction of the interval of the integral;

$$\begin{aligned}
&\mathrm{Re}\Big\langle \mathfrak{F}^{-1}[\chi_{\xi>0}\langle\cdot\rangle^{-s-1/2-\varepsilon}]\Big| \int_0^t U(\pm(t-t'))\big(U(\pm t')u_0\big)^2 dt'\Big\rangle_{H^s(\mathbb{R})}\\
&=\mathrm{Re}\int_0^\infty\int_0^t\int_0^\xi \exp(-it'(\sqrt{m^2+\xi^2}+\sqrt{m^2-(\xi-\eta)^2}-\sqrt{m^2+\eta^2}))dt'\\
&\qquad\cdot\langle\xi\rangle^{s-1/2-\varepsilon}\langle\xi-\eta\rangle^{-s-1/2-\varepsilon}\langle\eta\rangle^{-s-1/2-\varepsilon}\ d\eta\ d\xi.
\end{aligned}$$

For sufficiently small t and $0 \leq \xi \leq \eta$,

$$\mathrm{Re}\Big(\exp\big(-it'\big(\sqrt{m^2+\xi^2}-\sqrt{m^2+(\xi-\eta)^2}-\sqrt{m^2+\eta^2}\big)\big)\Big) \geq 1/2.$$

Then,

$$
\begin{aligned}
&\mathrm{Re}\left\langle \mathfrak{F}^{-1}[\chi_{\lceil 0,\infty)}]\langle\cdot\rangle^{-s-1/2-\varepsilon}, \int_0^t U(\pm(t-t'))\big(U(\pm t')u_0\big)^2 dt' \right\rangle \\
&\geq t/2 \int_0^\infty \int_\xi^\infty \langle\xi\rangle^{s-1/2-\varepsilon}\langle\xi-\eta\rangle^{-s-1/2-\varepsilon}\langle\eta\rangle^{-s-1/2-\varepsilon}\, d\eta\, d\xi \\
&\geq t/2 \int_1^\infty \int_\xi^{2\xi} \langle\xi\rangle^{-s-3/2-3\varepsilon} d\eta\, d\xi \\
&\gtrsim t \int_1^\infty \langle\xi\rangle^{-s-1/2-3\varepsilon} = \infty.
\end{aligned}
$$

for $\varepsilon \leq 1/6 - s/3$.

Chapter 3

Construction of Solutions for (SR) with a Priori Estimate

3.1 Introduction

In this chapter, we consider the Cauchy problems for the semirelativistic equation

$$\begin{cases} i\partial_t u \pm (m^2 - \Delta)^{1/2} u = \lambda |u|^{p-1} u, & t \in \mathbb{R},\ x \in \mathbb{R}, \\ \quad u(0) = u_0, & x \in \mathbb{R}, \end{cases} \tag{3.1.1}$$

with $m, \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, where $\partial_t = \partial/\partial t$ and Δ is the Laplacian in $\mathbb{R}$.

Here, we restate our main result.

Theorem 3.1.1. *Let p, $\lambda \in \mathbb{R}$, and u_0 satisfy one of the following:*

- $1 < p \leq 3$ *and* $\lambda \leq 0$,
- $1 < p < 3$ *and* $\lambda > 0$,
- $p = 3$, $\lambda > 0$, *and* $\|u\|_{L^2(\mathbb{R})} \ll 1$.

Then for any $u_0 \in H^{1/2}(\mathbb{R})$, there exists a global solution to (3.1.1). *Moreover, let $u_{0,n}$, $u_0 \in H^{1/2}(\mathbb{R})$ satisfy $u_{0,n} \to u_0$ in $H^{1/2}(\mathbb{R})$ as $n \to \infty$, and let u_n and u be the solutions of* (3.1.1) *with data $u_{0,n}$ and u_0, respectively. Then $u_n \to u$ in $L^\infty(-T, T; H^{1/2}(\mathbb{R}))$ for any $T > 0$ as $n \to \infty$.*

We prove Theorem 3.1.1 by a simple argument based on Yosida type approximation operator. We remark that Theorem 3.1.1 can be obtained by standard compactness argument. For details of the standard compactness argument, we refer the reader to [62]. On the other hand, in this chapter, we directly introduce a sequence of approximation solutions which converges the associated $H^{1/2}(\mathbb{R})$ valued solution in $L^\infty(-T, T; H^{1/2}(\mathbb{R}))$ for any $T > 0$.

We give a brief outline of this chapter. In Section 3.2, we collect some basic estimates for the proof of Theorem 3.1.1. In Section 3.3, we give a proof of Theorem 3.1.1.

3.2 Preliminary for the Proof of Theorem 3.1.1

In this section, we collect some basic estimates for the proof of Theorem 3.1.1.

The following three estimates are basic for the proof of Theorem 3.1.1

Lemma 3.2.1. *Let X be a Banach space such that $X \hookrightarrow X^*$, where X^* is the dual of X. Let $f, g \in C(\mathbb{R}; X) \cap C^1(\mathbb{R}; X^*)$. Then $\langle f, g\rangle \in C^1(\mathbb{R}; \mathbb{C})$ and*

$$\frac{d}{dt}\langle f(t), g(t)\rangle = \left\langle \frac{d}{dt} f(t) \middle| g(t) \right\rangle_X + \left\langle f(t) \middle| \frac{d}{dt} g(t) \right\rangle_X,$$

where $\langle \cdot \mid \cdot \rangle_X$ is the dual product for X and X^*.

Lemma 3.2.2 ([77])**.** *Let $2 \le p < \infty$. There exists $C > 0$ such that for any $\psi \in H^{1/2}(\mathbb{R})$,*

$$\|\psi\|_{L^p(\mathbb{R})} \le C\sqrt{p}\|\psi\|_{H^{1/2}(\mathbb{R})}.$$

Lemma 3.2.3 (Lemma 2.4)**.** *Let $r > 1$ and a, b, $T > 0$. Let $f : [0, T] \to [0, \infty)$ satisfy*

$$f(t) \le a + b \int_0^t f^{1-1/r}(t')dt'$$

for all $0 \le t \le T$. Then, $f(t) \le (a^{1/r} + br^{-1}t)^r$ for all $0 \le t \le T$.

We consider the following integral equation associated with (3.1.1) and corresponding approximation equation:

$$u(t) = U(\pm t)u_0 - i\lambda \int_0^t U(\pm(t - t')|u(t')|^{p-1}u(t')dt', \tag{3.2.1}$$

$$u_\rho(t) = U(\pm t)J_\rho u_0 - i\lambda \int_0^t U(\pm(t - t'))J_\rho\big(|J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')\big)dt', \tag{$3.2.1)_\rho$}$$

where J_ρ is approximation operator of Yosida type defined by $J_\rho = \mathfrak{F}^{-1}\rho^2(\rho^2 + \xi^2)^{-1}\mathfrak{F}$.

It is easily seen that (3.2.1) and $(3.2.1)_\rho$ has time-local solution in the $H^1(\mathbb{R})$ setting. If $u \in C([0, T]; H^1(\mathbb{R}))$ is a solution for (3.2.1), then $u \in C^1([0, T]; L^2(\mathbb{R}))$, since by the Sobolev embedding,

$$\||u|^{p-1}u\|_{L^2(\mathbb{R})} \le \|u\|^{p-1}_{L^\infty(\mathbb{R})}\|u\|_{L^2(\mathbb{R})} \lesssim \|u\|^p_{H^1(\mathbb{R})}.$$

and therefore, the Duhamel term of (3.2.1) is differentiable as $L^2(\mathbb{R})$ valued function. It is easily seen that $H^1(\mathbb{R})$ valued solutions for $(3.2.1)_\rho$ are also differentiable as $L^2(\mathbb{R})$ valued function. Then we have the following time-local well-posedness in $H^{1/2}(\mathbb{R})$ setting:

Lemma 3.2.4. *There exists a unique time-local solution to $(3.2.1)_\rho$ in $C([0, T); H^1(\mathbb{R}))$ for any $\rho > 0$ and $H^{1/2}(\mathbb{R})$ initial data.*

***proof*.** For any $s \in \mathbb{R}$ and $\rho \ge 1$

$$\begin{aligned}\|J_\rho f\|_{H^s(\mathbb{R})} &= \left\| \frac{\rho^2}{\rho^2 + .^2}\langle\cdot\rangle^s \hat{f} \right\|_{L^2(\mathbb{R})} \\ &\le \max(\rho, 1)^2 \|\langle\cdot\rangle^{s-2}\hat{f}\Big\|_{L^2(\mathbb{R})} \\ &\le \max(\rho, 1)^2\|f\|_{H^{s-2}(\mathbb{R})}.\end{aligned}$$

Therefore, The fact that solution map

$$\Phi_\pm(u) = U(\pm t)J_\rho u_0 - i\lambda \int_0^t U(\pm(t - t'))J_\rho\big(|J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')\big)dt'$$

is a contraction map in $C([0, T]; H^1(\mathbb{R}))$ follows from Lemma 1.3.14. Q.E.D.

Moreover, we have the following conservation laws.

Proposition 3.2.1. *Let $u_0 \in H^{1/2}(\mathbb{R})$. Let $u \in C(\mathbb{R}; H^{1/2}(\mathbb{R})) \cap C^1(\mathbb{R}; H^{-1/2}(\mathbb{R}))$ be a solution to the integral equation (3.2.1) for the initial data u_0. Then, $\|u(t)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}$ for any t.*

Let $\rho > 0$. Let $u_\rho \in C(\mathbb{R}; H^{1/2}(\mathbb{R})) \cap C^1(\mathbb{R}; H^{-1/2}(\mathbb{R}))$ be a solution to the integral equation $(3.2.1)_\rho$ for the initial data $J_\rho u_0$. Then, $\|u_\rho(t)\|_{L^2(\mathbb{R})} = \|J_\rho u_0\|_{L^2(\mathbb{R})}$ for any t.

proof. The following formal calculations are justified by extending $L^2(\mathbb{R})$ scalar product to $H^{-1/2}(\mathbb{R})$-$H^{1/2}(\mathbb{R})$ duality:

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R})} &= 2\mathrm{Re}\langle \partial_t u(t) \mid u(t)\rangle_{H^{1/2}(\mathbb{R})} = 2\mathrm{Im}\langle i\partial_t u(t) \mid u(t)\rangle_{H^{1/2}(\mathbb{R})} \\
&= 2\mathrm{Im}\langle \mp(m^2-\Delta)^{1/2}\ u(t) + \lambda|u(t)|^{p-1}u(t) \mid u(t)\rangle_{H^{1/2}(\mathbb{R})} \\
&= 2\mathrm{Im}\|u(t)\|_{L^{p+1}(\mathbb{R})} = 0, \\
\frac{d}{dt}\|u_\rho(t)\|^2_{L^2(\mathbb{R})} &= 2\mathrm{Re}\langle \partial_t u_\rho(t) \mid u_\rho(t)\rangle_{H^{1/2}(\mathbb{R})} = 2\mathrm{Im}\langle i\partial_t u_\rho(t) \mid u_\rho(t)\rangle_{H^{1/2}(\mathbb{R})} \\
&= 2\mathrm{Im}\langle \mp(m^2-\Delta)^{1/2}\ u_\rho(t) + \lambda J_\rho|J_\rho u(t)|^{p-1}J_\rho u(t) \mid J_\rho u(t)\rangle_{H^{1/2}(\mathbb{R})} \\
&= 2\mathrm{Im}\|J_\rho u(t)\|_{L^{p+1}(\mathbb{R})} = 0,
\end{aligned}$$

where we used the following identity:

$$((m^2-\Delta)^{1/2}f \mid f) = ((m^2+\cdot^2)^{1/2}\hat{f} \mid \hat{f}) = \|(m^2+\cdot)^{1/4}\hat{f}\|^2_{L^2(\mathbb{R})}.$$

Q.E.D.

Proposition 3.2.2. *Let $\lambda \in \mathbb{R}$ and $u_0 \in H^{1/2}(\mathbb{R})$. Let $u \in C(\mathbb{R}; H^1(\mathbb{R})) \cap C^1(\mathbb{R}; L^2(\mathbb{R}))$ be a solution to the integral equation (3.2.1) for the initial data u_0. Then $E(u(t)) = E(u_0)$ for any t, where*

$$E(f) = \|(m^2-\Delta)^{1/4}f\|^2_{L^2(\mathbb{R})} - \frac{\lambda}{p+1}\|f\|^{p+1}_{L^{p+1}(\mathbb{R})}.$$

Let $\rho > 0$. Let $u_\rho \in C(\mathbb{R}; H^1(\mathbb{R})) \cap C^1(\mathbb{R}; L^2(\mathbb{R}))$ be a solution to the integral equation $(3.2.1)_\rho$ for the initial data $J_\rho u_0$. Then, $E_\rho(u_\rho(t)) = E_\rho(J_\rho u_0)$ for any t, where

$$E_\rho(f) = \|(m^2-\Delta)^{1/4}f\|^2_{L^2(\mathbb{R})} - \frac{\lambda}{p+1}\|J_\rho f\|^{p+1}_{L^{p+1}(\mathbb{R})}.$$

proof. If $u \in C(\mathbb{R}; H^1(\mathbb{R})) \cap C^1(\mathbb{R}; L^2(\mathbb{R}))$, then the squared norms $\|(m^2-\Delta)^{1/4}u\|^2_{L^2(\mathbb{R})}$ is differentiable and we have

$$\begin{aligned}
\frac{d}{dt}\|(m^2-\Delta)^{\frac{1}{4}}u(t)\|^2_{L^2(\mathbb{R})} &= 2\mathrm{Re}(\mp i\partial_t u(t) \pm \lambda|u(t)|^{p-1}u(t) \mid \pm\partial_t u(t)) \\
&= \frac{\lambda}{p+1}\frac{d}{dt}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R})}, \\
\frac{d}{dt}\|(m^2-\Delta)^{\frac{1}{4}}u_\rho(t)\|^2_{L^2(\mathbb{R})} &= 2\mathrm{Re}(\mp i\partial_t u_\rho(t) \pm \lambda J_\rho|J_\rho u_\rho(t)|^{p-1}J_\rho u_\rho(t) \mid \pm\partial_t u_\rho(t)) \\
&= \frac{\lambda}{p+1}\frac{d}{dt}\|J_\rho u_\rho(t)\|^{p+1}_{L^{p+1}(\mathbb{R})}.
\end{aligned}$$

Q.E.D.

Then, we have the following $\dot{H}^{1/2}(\mathbb{R})$ boundedness of solutions to $(3.2.1)_\rho$. Therefore, $H^{1/2}(\mathbb{R})$ valued time-local solutions of $(3.2.1)_\rho$ are extended globally in time.

Lemma 3.2.5. *For any $u_0 \in H^{1/2}(\mathbb{R})$ and $T > 0$, the associated solution $u_\rho \in C([0,T); H^{1/2}(\mathbb{R})$ to $(3.2.1)_\rho$ satisfies that if $1 < p \leq 3$ and $\lambda \leq 0$, then*

$$\|u_\rho\|_{L^\infty(0,T;\dot{H}^{1/2}(\mathbb{R}))} \leq E_\rho(J_\rho u_0),$$

if $1 < p < 3$ and $\lambda > 0$, then

$$\|u_\rho\|^2_{L^\infty(0,T;\dot{H}^{1/2}(\mathbb{R}))} \leq 2E_\rho(J_\rho u_0) + \left(\frac{2\lambda C_{g,p+1}}{p+1}\|u_0\|^2_{L^2(\mathbb{R})}\right)^{\frac{2}{3-p}},$$

and if $p = 3$, $\lambda > 0$, and $\|u_0\|_{L^2(\mathbb{R})} < 2\lambda^{-1/2}C_{g,4}^{-2}$, then

$$\|u_\rho\|^2_{L^\infty(0,T;\dot{H}^{1/2}(\mathbb{R}))} \leq \frac{E_\rho(J_\rho u_0)}{1 - \frac{\lambda C_{g,4}^4}{4}\|u_0\|^2_{L^2(\mathbb{R})}},$$

where $C_{g,p}$ is a best constant of the the Gagliardo-Nirenberg inequality

$$\|f\|_{L^p(\mathbb{R})} \leq C_{g,p}\|f(t)\|^{\frac{2}{p}}_{L^2(\mathbb{R})}\|f(t)\|^{\frac{p-2}{p}}_{\dot{H}^{1/2}(\mathbb{R})}.$$

proof. If $\lambda < 0$, then Propositions 3.2.2, for any $t \in [0,T)$,

$$\|u_\rho(t)\|_{\dot{H}^{1/2}(\mathbb{R})} \leq E_\rho(u_\rho(t)) = E_\rho(J_\rho u_0).$$

If $1 < p < 3$ and $\lambda > 0$, for any $t \in [0,T)$,

$$\begin{aligned}\|u_\rho(t)\|^2_{\dot{H}^{1/2}(\mathbb{R})} &\leq \|(m^2 - \Delta)^{1/2}u_\rho(t)\|^2_{L^2(\mathbb{R})} \\ &\leq E_\rho(u_\rho(t)) + \frac{\lambda}{p+1}\|J_\rho u_\rho(t)\|^{p+1}_{L^{p+1}(\mathbb{R})} \\ &\leq E_\rho(J_\rho u_0) + \frac{\lambda C^{p+1}_{g,p+1}}{p+1}\|u_0\|^2_{L^2(\mathbb{R})}\|u_\rho(t)\|^{p-1}_{\dot{H}^{1/2}(\mathbb{R})}.\end{aligned}$$

Therefore,

$$\|u_\rho\|^2_{L^\infty(0,T;\dot{H}^{1/2}(\mathbb{R}))} \leq 2E_\rho(J_\rho u_0) + \left(\frac{2\lambda C^{p+1}_{g,p+1}}{p+1}\|u_0\|^2_{L^2(\mathbb{R})}\right)^{\frac{2}{3-p}}.$$

Moreover, if $p = 3$, $\lambda > 0$, and $\|u_0\|_{L^2(\mathbb{R})} < 2\lambda^{-1/2}C_{g,4}^{-2}$, then

$$\|u_\rho\|^2_{L^\infty(0,T;\dot{H}^{1/2}(\mathbb{R}))} \leq \frac{E_\rho(J_\rho u_0)}{1 - \frac{\lambda C_{g,4}^4}{4}\|u_0\|^2_{L^2(\mathbb{R})}}.$$

Q.E.D.

By using the energy conservation, we also obtain the following continuity lemma.

Lemma 3.2.6. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty(\mathbb{R}; H^{1/2}(\mathbb{R}))$ which converges to $f \in L^\infty(\mathbb{R}, H^{1/2}(\mathbb{R}))$ in $L^2(\mathbb{R})$ as $n \to \infty$ locally uniformly. If $(E(f_n))_{n\in\mathbb{N}}$ converges to $E(f)$ as $n \to \infty$ locally uniformly, then $\|f(t) - f_n(t)\|_{H^{1/2}(\mathbb{R})} \to 0$ locally uniformly.*

proof. Since $H^{1/2}(\mathbb{R})$ is a Hilbert space,

$$\|f(t) - f_n(t)\|^2_{H^{1/2}(\mathbb{R})} = 2\mathrm{Re}((1-\Delta)^{1/4}f(t) \mid (1-\Delta)^{1/4}(f(t) - f_n(t))) - \|f(t)\|^2_{H^{1/2}(\mathbb{R})} + \|f_n(t)\|^2_{H^{1/2}(\mathbb{R})}.$$

Since $(f_n)_{n\in\mathbb{N}}$ converges f in $L^2(\mathbb{R})$ as $n \to \infty$ locally uniformly, $2\mathrm{Re}((1-\Delta)^{1/4}f(t) \mid (1-\Delta)^{1/4}(f(t) - f_n(t)))$ also goes to 0 locally uniformly. Moreover, since $(E(f_n))_{n\in\mathbb{N}}$ converges to $E(f)$ as $n \to \infty$ locally uniformly, $(\|f_n(\cdot)\|_{H^{1/2}(\mathbb{R})})_{n\in\mathbb{N}}$ converges to $\|f(\cdot)\|_{H^{1/2}(\mathbb{R})}$ as $n \to \infty$ locally uniformly, Q.E.D.

3.3 Proof of Theorem 3.1.1

In this section, we divide the proof of Theorem 3.1.1 into two parts: proof of the existence of solutions and proof of the continuity in time and the continuous dependence on initial data.

3.3.1 Proof of Existence of Solutions

Let $u_0 \in H^{1/2}(\mathbb{R})$. By Lemma 3.2.5 and Proposition 3.2.1, there exists $M = M(\lambda, p, u_0)$ such that

$$\sup_{\rho>0} \|u_\rho\|_{L^\infty(\mathbb{R};H^{1/2}(\mathbb{R}))} \le M.$$

For $t \in \mathbb{R}$, we estimate

$$\begin{aligned}
&\|u_\rho(t) - u_\sigma(t)\|_{L^2(\mathbb{R})} \\
&\le \|(J_\rho - J_\sigma)u_0\|_{L^2(\mathbb{R})} + |\lambda| \int_0^t \big\|(1-J_\rho)\big(|J_\rho u_\rho(t')|^{p-1} J_\rho u_\rho(t')\big)\big\|_{L^2(\mathbb{R})} dt' \\
&+ |\lambda| \int_0^t \big\|(1-J_\sigma)\big(|J_\sigma u_\sigma(t')|^{p-1} J_\sigma u_\sigma(t')\big)\big\|_{L^2(\mathbb{R})} dt' \\
&+ |\lambda| \int_0^t \big\||J_\rho u_\rho(t')|^{p-1} J_\rho u_\rho(t') - |J_\sigma u_\sigma(t')|^{p-1} J_\sigma u_\sigma(t')\big\|_{L^2(\mathbb{R})} dt'.
\end{aligned} \tag{3.3.1}$$

For sufficiently large r, by the Hölder, Gagliardo-Nirenberg inequalities, and Lemma 3.2.2,

$$\begin{aligned}
&\||J_\rho u_\rho(t')|^{p-1} J_\rho u_\rho(t') - |J_\sigma u_\sigma(t')|^{p-1} J_\sigma u_\sigma(t')\|_{L^2(\mathbb{R})} \\
&\le C\|(|J_\rho u_\rho(t')|^{p-1} + |J_\sigma u_\sigma(t')|^{p-1})(J_\rho u_\rho(t') - J_\sigma u_\sigma(t'))\|_{L^2(\mathbb{R})} \\
&\le C\big(\|J_\rho u_\rho(t')\|^{p-1}_{L^{4(p-1)}(\mathbb{R})} + \|J_\sigma u_\sigma(t')\|^{p-1}_{L^{4(p-1)}(\mathbb{R})}\big) \\
&\cdot \big(\|(1-J_\rho)u_\rho(t')\|_{L^4(\mathbb{R})} + \|(1-J_\sigma)u_\sigma(t')\|_{L^4(\mathbb{R})}\big) \\
&+ C\big(\|J_\rho u_\rho(t')\|^{p-1}_{L^{2r(p-1)}(\mathbb{R})} + \|J_\sigma u_\sigma(t')\|^{p-1}_{L^{2r(p-1)}(\mathbb{R})}\big)\|u_\rho(t') - u_\sigma(t')\|_{L^{2r'}(\mathbb{R})} \\
&\le CM^{p-1}\big(\|(1-J_\rho)u_\rho(t')\|_{H^{1/4}(\mathbb{R})} + \|(1-J_\sigma)u_\sigma(t')\|_{H^{1/4}(\mathbb{R})}\big) \\
&+ Cr^{(p-1)/2} M^{p-1+1/r} \|u_\rho(t') - u_\sigma(t')\|^{1/r'}_{L^2(\mathbb{R})},
\end{aligned} \tag{3.3.2}$$

where C is independent of r. Since

$$\frac{\xi^2}{\rho^2 + \xi^2} \le \frac{|\xi|^{1/4}}{\rho^{1/4}},$$

we have

$$\|(1-J_\rho)u_\rho(t')\|_{H^{1/4}(\mathbb{R})} \le \rho^{-1/4}\|u_\rho(t')\|_{H^{1/2}(\mathbb{R})} \le M\rho^{-1/4}. \tag{3.3.3}$$

Similarly,

$$\begin{aligned}
&\|(1-J_\rho)(|J_\rho u_\rho(t')|^{p-1} J_\rho u_\rho(t'))\|_{L^2(\mathbb{R})} \\
&\le \rho^{-1/4}\||J_\rho u_\rho(t')|^{p-1} J_\rho u_\rho(t')\|_{\dot{H}^{1/4}(\mathbb{R})} \\
&\le C\rho^{-1/4}\||J_\rho u_\rho(t')|^{p-1} J_\rho u_\rho(t')\|_{\dot{B}^{1/4}_{2,2}(\mathbb{R})}.
\end{aligned} \tag{3.3.4}$$

By Lemma 1.3.14 and the Sobolev embedding,

$$\begin{aligned}
\||J_\rho u_\rho(t')|^{p-1} J_\rho u_\rho(t')\|_{\dot{B}^{1/4}_{2,2}(\mathbb{R})} &\le C\|J_\rho u_\rho(t')\|_{\dot{B}^{1/4}_{4,2}(\mathbb{R})}\|J_\rho u_\rho(t')\|^{p-1}_{L^{4(p-1)}(\mathbb{R})} \\
&\le CM^p
\end{aligned} \tag{3.3.5}$$

with some positive constant C. Combining (3.3.1), (3.3.2), (3.3.3), and (3.3.5),

$$\begin{aligned}&\|u_\rho(t)-u_\sigma(t)\|_{L^2(\mathbb{R})}\\&\leq C(\rho^{-1/4}+\sigma^{-1/4})+Cr^{(p-1)/2}\int_0^t \|u_\rho(t')-u_\sigma(t')\|_{L^2(\mathbb{R})}^{1-1/r}dt'.\end{aligned} \tag{3.3.6}$$

By (3.3.6) and Lemma 3.2.3,

$$\|u_\rho(t)-u_\sigma(t)\|_{L^2(\mathbb{R})}\leq C\big((\rho^{-1/4}+\sigma^{-1/4})^{1/r}+r^{(p-3)/2}t\big)^r.$$

For any $\varepsilon>0$, let $r>\log_2(1/\varepsilon)$, $t<\log_2(1/\varepsilon)^{(3-p)/2}$, $\rho,\sigma>2^{4+2/(\log_2(1/\varepsilon)}$,

$$\big((\rho^{-1/4}+\sigma^{-1/4})^{1/r}+r^{(p-3)/2}t\big)^r\leq 2^{\log_2\varepsilon}=\varepsilon.$$

This shows that u_ρ is a Cauchy net in $L^\infty(-1/4,1/4;L^2(\mathbb{R}))$. By repeating this argument, u_ρ is shown to be a Cauchy net in $L^\infty(-T,T;L^2(\mathbb{R}))$ for any $T>0$ and therefore $(u_\rho)_{\rho\geq 0}$ converges locally uniformly in time. Let $u\in L^\infty(\mathbb{R},H^{1/2})$ be the $L^2(\mathbb{R})$ limit of u_ρ. Since $\|u_\rho\|_{L^\infty(\mathbb{R};H^{1/2}(\mathbb{R}))}\leq M$ for any $\rho\geq 0$, by Lemma 1.3.2, u is also estimated by $\|u\|_{L^\infty(\mathbb{R};H^{1/2}(\mathbb{R}))}\leq M$. Then we have

$$\begin{aligned}&\Big\|\int_0^t U(\pm(t-t'))J_\rho\Big(|J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')\Big)dt'\\&-\int_0^t U(\pm(t-t'))|u(t')|^{p-1}u(t')dt'\Big\|_{L^2(\mathbb{R})}\\&\leq\int_0^t\Big\|(1-J_\rho)\Big(|J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')\Big)\Big\|_{L^2(\mathbb{R})}dt'\\&+\int_0^t\Big\||J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')-|u(t')|^{p-1}u(t')\Big\|_{L^2(\mathbb{R})}dt'.\end{aligned}$$

For any $0\leq t'\leq t$, $\|(1-J_\rho)(|J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t'))\|_{L^2(\mathbb{R})}$ goes to 0 as $\rho\to\infty$ by (3.3.4) and (3.3.5). Moreover, by a similar calculation to (3.3.2), it is shown that

$$\begin{aligned}&\||J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')-|u(t')|^{p-1}u(t')\|_{L^2(\mathbb{R})}\\&\leq C+Cr^{(p-1)/2}\|u_\rho(t')-u(t)\|_{L^2(\mathbb{R})}^{1-1/r}\end{aligned}$$

holds. Then, by the Lebesgue dominated convergence theorem,

$$\begin{aligned}&\Big\|\int_0^t U(\pm(t-t'))J_\rho\Big(|J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')\Big)dt'\\&-\int_0^t U(\pm(t-t'))|u(t')|^{p-1}u(t')dt'\Big\|_{L^2(\mathbb{R})}\\&\leq\int_0^t\Big\|(1-J_\rho)\Big(|J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')\Big)\Big\|_{L^2(\mathbb{R})}dt'\\&\to 0\end{aligned}$$

for any $t\in\mathbb{R}$ as $\rho\to\infty$. This means, each of the terms of $(3.2.1)_\rho$ converges those of (3.2.1) in $L^2(\mathbb{R})$ for any t and therefore, $u\in L^\infty(\mathbb{R};H^{1/2}(\mathbb{R}))$ is a solution to (3.1.1).

3.3.2 Proof of the Continuity of Solutions

In this subsection, we prove that energy conservation and continuity in time of $H^{1/2}(\mathbb{R})$ valued solution, and the continuous dependence of solutions on initial data in the energy space $H^{1/2}(\mathbb{R})$.

Let $u_{1,0}$ and $u_{2,0} \in H^{1/2}(\mathbb{R})$. Let u_1 and $u_2 \in L^\infty(\mathbb{R}; H^{1/2}(\mathbb{R}))$ be the solutions of (3.1.1) with initial data $u_{1,0}$ and $u_{2,0}$, respectively. Then, by the same argument as in subsection 3.3.1, for any $t \in \mathbb{R}$ and $p > 2$,

$$\|u_1(t) - u_2(t)\|_{L^2(\mathbb{R})} \lesssim \|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})} + p \int_0^t \|u_1(t') - u_2(t')\|_{L^2(\mathbb{R})}^{1-1/p} dt'.$$

Then by Lemma 3.2.3 we obtain

$$\|u_1(t) - u_2(t)\|_{L^2} \lesssim \left(\|u_{1,0} - u_{2,0}\|_{L^2}^p + t\right)^{1/p}.$$

This shows that the solutions of (3.1.1) for $H^{1/2}(\mathbb{R})$ initial data is unique and that the solutions depends on initial data continuously in $L^2(\mathbb{R})$ locally uniformly in time.

Next, we show the energy conservation. Let $u_0 \in H^{1/2}(\mathbb{R})$ and $u \in L^\infty(\mathbb{R}; H^{1/2}(\mathbb{R}))$ be the solution of (3.1.1) for the initial data u_0. Let $\rho > 0$ and $u_\rho \in L^\infty(\mathbb{R}; H^{1/2})$ be the solutions of $(3.2.1)_\rho$ for the initial data $J_\rho u_0$. By Lemma 1.3.2, for any $t \in \mathbb{R}$,

$$E(u(t)) \leq \liminf_{\rho \to \infty} E_\rho(u_\rho(t)) = \liminf_{\rho \to \infty} E_\rho(J_\rho u_0) \leq E(u_0).$$

Moreover, since u is the unique solution, the solution for the initial data $u(t)$ coincides with $u(\cdot + t)$. Then we obtain the inverse inequality by the same argument. This shows the energy conservation.

Since $H^{1/2}(\mathbb{R})$ valued solution of (3.1.1) are in $C(\mathbb{R}; L^2(\mathbb{R}))$, they are also continuous in $H^{1/2}(\mathbb{R})$ by the energy conservation and Lemma 3.2.6. Similarly to the continuously dependence of solutions in $L^2(\mathbb{R})$, it is shown that an $H^{1/2}(\mathbb{R})$ valued solution also continuously depends on the initial data in $H^{1/2}(\mathbb{R})$ for each $t \in \mathbb{R}$. Since an $H^{1/2}(\mathbb{R})$ valued solution is in $C(\mathbb{R}; H^{1/2}(\mathbb{R}))$, by Lemma 3.2.6 again, $H^{1/2}(\mathbb{R})$ valued solutions continuously depend on the initial data in $H^{1/2}(\mathbb{R})$ locally uniformly in time.

Chapter 4

Nonexistence of Solutions to (SR) without Gauge Invariance

4.1 Introduction

In this chapter, we consider the Cauchy problem for the semirelativistic equations

$$\begin{cases} i\partial_t u \pm (m^2-\Delta)^{1/2}u = \lambda|u|^p, & t\in\mathbb{R},\ x\in\mathbb{R}, \\ u(0)=u_0, & x\in\mathbb{R}, \end{cases} \tag{4.1.1}$$

with $\lambda \in \mathbb{C}\backslash\{0\}$ and $m\in\mathbb{R}$.

Here, we are interested in the local solvability of the Cauchy problem of (4.1.1). In general spacial dimension n, by the standard contraction argument, we have the unique local solution to (4.1.1) for $s > n/2$ and $u_0 \in H^s(\mathbb{R}^n)$. Moreover, (4.1.1) is expected to have a local solution for any $H^s(\mathbb{R}^n)$ initial data with $s > s_{1,p}^{(SR)}$, where $s_{1,p}^{(SR)}$ is defined as (1.4.1). However, in Chapter 2, it is shown that for $n = 1$, $p = 2$, and $s < 1/2$, the solution map (4.1.1) is not C^2 in $H^s(\mathbb{R})$. This means that it is impossible to obtain local solution to (4.1.1) by an iteration argument. In this chapter, we discuss about (4.1.1) further from a negative standpoint and we show the sharp criteria of the smoothness of initial data so that for any $H^s(\mathbb{R})$ initial data and for sufficiently small $T > 0$, we have a time-local solution to (4.1.1) in $C([0,T), H^s(\mathbb{R}))$.

Nonexistence results for local and global solutions have been obtained by test function method which is introduced by Zhang in [93, 94]. There is a large literature on test function method and we refer the reader to [52, 53, 54, 55]. Test function method is a method to deny the existence of weak solutions by showing a contradiction of weak equations with a sequence of test functions. To apply test function method to (4.1.1), however, a serious difficulty arises when we try to handle the non-local operator $(m^2-\Delta)^{1/2}$. It is because in order to show a contradiction of a weak equation, we need to cancel unknown weak solutions in the weak equation. In order to cancel weak solutions, the positivity of nonlinearity and pointwise estimate of test functions are required but it seems difficult to estimate a test function with $(m^2-\Delta)^{1/2}$ pointwisely, since $(m^2-\Delta)^{1/2}$ is non-local. To overcome this difficulty, we apply $-\mathrm{Im}\overline{\lambda}(i\partial_t \mp (m^2-\Delta)^{1/2})$ to (4.1.1) to obtain

$$\Box \mathrm{Im}(\overline{\lambda}u) + m^2\mathrm{Im}(\overline{\lambda}u) = \partial_t^2\mathrm{Im}(\overline{\lambda}u) - \Delta\mathrm{Im}(\overline{\lambda}u) + m^2\mathrm{Im}(\overline{\lambda}u) = -|\lambda|^2\partial_t|u|^p. \tag{4.1.2}$$

We remark that this transformation is a modified derivation of Klein-Gordon equation from semirelativistic equation in Section 1.2. In Section 4.2, we revisit the transformation of (4.1.1).

By (4.1.2) and the associated test function method, it can be shown that there exists no global solutions to (4.1.1) with $m = 0$ and $n = 1$ for $1 < p \le 2$. In [55], Inui shows that the large data blow-up for $s \ge s_{1,p}^{(SR)}$ and nonexistence of local solutions for $s < s_p$ to (4.1.1) with $n \ge 1$ and $m \in \mathbb{R}$ by improving test function method for (4.1.2). We also remark that similar nonexistence results are obtained for the Cauchy problem of nonlinear Schrödinger equations

$$i\partial_t u + \Delta u = \lambda |u|^p$$

in the case of $\mathbb{R}^n$ by Inui and Ikeda in [52, 53] and Ikeda and Wakasugi in [54] and in the case of $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ by Oh in [78]. We remark that in earlier works above, in the case of $\mathbb{R}^n$, the non-existence of solutions are argued by scaling criticality. But in this chapter, we show that the non-existence of solutions to (4.1.1) is obtained even in scaling subcritical case.

To restate our main result, we reintroduce the definition of time-local weak solutions of (4.1.1). For $T > 0$, we define function spaces A and A_T as follows:

$$A = C([0,\infty); H^2(\mathbb{R};\mathbb{R})) \cap C^1([0,\infty); H^1(\mathbb{R};\mathbb{R})),$$
$$A_T = \{\psi \in X \mid \operatorname{supp}\psi \subset (-\infty, T) \times \mathbb{R}\}.$$

Let $(\cdot \mid \cdot)$ be the usual L^2 scalar product defined by $(f \mid g) = \int f\overline{g}$. Then we define weak local solutions to (4.1.1).

Definition 4.1.1. *Let $T > 0$ and $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$. We say that u is a weak time-local solution to* (4.1.1), *if u belongs to $L^1_{\mathrm{loc}}(0,T; L^2(\mathbb{R}) \cap L^p(\mathbb{R}))$ and the following identity*

$$\int_0^T \big(u(t)\big|i\partial_t\psi(t) \pm (m^2-\Delta)^{1/2}\psi(t)\big)dt = i(u_0|\psi(0)) + \lambda\int_0^T \big(|u(t)|^p\big|\psi(t)\big)dt \tag{4.1.3}$$

holds for any $\psi \in A_T$, where the double-sign corresponds to the sign of (4.1.1).

Then we restate our main result below:

Theorem 4.1.1. *Let $1 < p < \infty$ and let $f \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ satisfy*

$$\exists\delta > 0 \text{ s.t. } f > 0 \text{ on } (-\delta,\delta) \text{ and } f \text{ is decreasing on } (0,\delta), \tag{4.1.4}$$
$$\lim_{\varepsilon \searrow 0} f(\varepsilon) = \infty. \tag{4.1.5}$$

Then there exists no $T > 0$ such that there exists a local weak solution to (4.1.1) *with $u_0 = -i\overline{\lambda}^{-1}f$.*

In Remark 1.5.4, It is shown that there exists $f \in H^{1/2}(\mathbb{R})$ such that f satisfies (4.1.4) and (4.1.5). Since $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ with $s > 1/2$, this means that $H^{1/2}(\mathbb{R})$ is the threshold so that for any $H^s(\mathbb{R})$ initial data and some $T > 0$, we have a solution in $C([0,T), H^s(\mathbb{R})) \cap C^1([0,T), H^{s-1}(\mathbb{R}))$ to the Cauchy problem of (4.1.1).

In Section 4.3, We give a proof of Theorem 4.1.1. The difficulty to prove Theorem 4.1.1 is the construction of a sequence of test functions to obtain the nonexistence results in scaling subcritical case. Since the sequence of test functions introduced by Zhang in [93, 94] is constructed by the scaling transformation under which (4.1.1) is invariant, it seems impossible to obtain the nonexistence results in scaling subcritical case with his test functions. To overcome this difficulty, we cancel the second derivatives of test functions and break the balance of scaling for test functions. In particular, we use a test function of the form

$$\psi(t,x) = \phi_1(t+x)\phi_2(t-x).$$

A direct calculation gives

$$\Box\psi(t,x) = 4\phi_1'(t+x)\phi_2'(t-x)$$

and ψ allows us to scale only ϕ_2 without any loss. We remark that with a test function of the form

$$\psi(t,x) = \phi_1(t)\phi_2(x),$$

scaling only ϕ_2 causes a loss and the nonexistence result of [55] seems to be optimal from the view point of the scaling criticality. In Section 4.3, we give a detailed proof of Theorem 4.1.1 with this idea.

4.2 Preliminary

In this section, we revisit the modification of (4.1.1) in order to apply a test function method.

The corresponding local weak solutions to (4.1.2) are defined as follows:

Definition 4.2.1. *Let $T > 0$ and $u_0 \in L^1_{loc}(\mathbb{R})$. We say that u is a weak time-local solution to* (4.1.2)*, if u belong to $L^1_{\mathrm{loc}}(0,T;L^2(\mathbb{R}) \cap L^p(\mathbb{R}))$ and the following identity*

$$\begin{aligned}
&\int_0^T \big(\mathrm{Im}(\overline{\lambda}u)(t)\big|\Box\psi(t) + m^2\psi(t)\big)dt \\
&= \pm\big(\mathrm{Re}(\overline{\lambda}u_0)\big|(m^2-\Delta)^{1/2}\psi(0)\big) + \big(\mathrm{Re}(i\overline{\lambda}u_0)\big|\partial_t\psi(0)\big) \\
&+ |\lambda|^2 \int_0^T \big(|u(t)|^p\big|\partial_t\psi(t)\big)dt
\end{aligned} \tag{4.2.1}$$

holds for $\psi \in C^2(\mathbb{R}^2;\mathbb{R})$ with $\operatorname{supp}\psi \subset (-\infty,T)\times\mathbb{R}$, where the double-sign corresponds to the sign of (4.1.1)*.*

Weak time-local solutions to (4.1.1) are shown to be those to (4.1.2) as follows:

Lemma 4.2.1. *Let $u_0 \in L^2(\mathbb{R})$. Then, time-local weak solutions to* (4.1.1) *are those to* (4.1.2)*.*

proof. Let $\phi \in C^2(\mathbb{R}^2;\mathbb{R})$. Then $(-\Delta)^{1/2}\phi$ and $\partial_t\phi$ belong to A. By taking real and imaginary parts of (4.1.3) with ψ replaced by $\lambda(-\Delta)^{1/2}\phi$ and $\lambda\partial_t\phi$, respectively, we obtain

$$\begin{aligned}
&\mathrm{Re}\int_0^\infty \big(\overline{\lambda}u(t)\big|i\partial_t^2\phi(t) \pm \partial_t(-\Delta)^{1/2}\phi(t)\big)dt \\
&= \int_0^\infty \big(v(t)\big|\partial_t^2\phi(t)\big)dt \pm \int_0^\infty \big(\mathrm{Re}(\overline{\lambda}u(t))\big|\partial_t(-\Delta)^{1/2}\phi(t)\big)dt \\
&= -(v(0)|\partial_t\phi(0)) + |\lambda|^2\int_0^\infty \big(|u(t)|^p\big|\partial_t\phi(t)\big)dt, \\
&\mathrm{Im}\int_0^\infty \big(\overline{\lambda}u(t)\big|i\partial_t(-\Delta)^{1/2}\phi(t) \mp \Delta\phi(t)\big)dt \\
&= -\int_0^\infty \big(\mathrm{Re}(\overline{\lambda}u(t))\big|\partial_t(-\Delta)^{1/2}\phi(t)\big)dt \mp \int_0^\infty \big(v(t)\big|\Delta\phi(t)\big)dt \\
&= \big(\mathrm{Re}(\overline{\lambda}u_0)\big|(-\Delta)^{1/2}\phi(0)\big).
\end{aligned}$$

By combining those identities, we obtain (4.2.1). Q.E.D.

Remark 4.2.2. *For $L^2(\mathbb{R})$ initial data and $T > 0$, solutions to* (4.1.1) *which belong to $L^1_{\mathrm{loc}}(0,T;L^p(\mathbb{R})) \cap C([0,T);L^2(\mathbb{R})) \cap C^1([0,T),H^{-1}(\mathbb{R}))$ are also shown to satisfy* (4.1.3) *and therefore they are also weak time-local solutions to* (4.1.2)*.*

4.3 Proof of Theorem 4.1.1

For simplicity, we take $\lambda = 1$. Assume that for $0 < T < T' < \min(1, \delta)$, there is a local weak solution to (4.1.1) on $[0, T']$. Let $\phi \in C^\infty(\mathbb{R}; [0, 1])$ satisfy

$$\phi(y) = \begin{cases} 1 & \text{if} \quad y < 0, \\ \searrow & \text{if} \quad 0 \le y \le T, \\ 0 & \text{if} \quad y > T. \end{cases}$$

Let $0 < \rho < 1$ and let $\phi_\rho(y) = \phi(y/\rho)$, $\phi'_\rho(y) = \phi'(y/\rho)$. Let $l \in \mathbb{Z}$ satisfy $l \ge p'$ and let

$$\psi_\rho(t, x) = -\phi(t - x)^l \phi_\rho(t + x)^l.$$

Then

$$\operatorname{supp} \psi_\rho \subset (-\infty, T] \times [-\min(1, \delta), \min(1, \delta)].$$

The first term of the right hand side of (4.2.1) is canceled since f is real-valued, and we estimate other terms on the right hand side of (4.2.1) by (4.1.4) as follows:

$$\begin{aligned}
&(\mathrm{Re}(iu_0) \mid \partial_t \psi_\rho(0)) \\
&\ge \rho^{-1} l \int_0^{T\rho} f(x) |\phi'_\rho(x)| \phi_\rho(x)^{l-1} dx \\
&\ge f(\rho\delta), \\
&\int_0^T (|u(t)|^p \mid \partial_t \psi_\rho(t, x)) dt \\
&\ge \rho^{-1} l \int_0^T (|u(t)|^p \mid |\phi'_\rho(t + x)| \phi_\rho(t + x)^{l-1} \phi(t - x)^l) dt \\
&= \rho^{-1} l \| u(t) |\phi'_\rho(t + x)|^{1/p} \phi_\rho(t + x)^{(l-1)/p} \phi(t - x)^{l/p} \|^p_{L^p([0,T]\times\mathbb{R})}.
\end{aligned}$$

Let

$$I = \| u(t) |\phi'_\rho(t + x)|^{1/p} \phi_\rho(t + x)^{(l-1)/p} \phi(t - x)^{l/p} \|_{L^p([0,T]\times\mathbb{R})}.$$

By the Hölder and Young inequalities,

$$\begin{aligned}
&\left| \int_0^T (\mathrm{Im}(\overline{\lambda} u)(t) \mid m^2 \psi_\rho(t)) dt \right| \le m^2 2^{1/p'} \| u(t) \|_{L^1([0,T]; L^p([-1,1]))}, \\
&\left| \int_0^T (\mathrm{Im}(\overline{\lambda} u)(t) \mid \Box \psi_\rho(t)) dt \right| \\
&\le 4\rho^{-1} l^2 \int_0^T \left| (\mathrm{Im}(\overline{\lambda} u)(t) \mid \phi'(t - x) \phi'_\rho(t + x) \phi(t - x)^{l-1} \phi_\rho(t + x)^{l-1}) \right| dt \\
&\le 4\rho^{-1} l^2 \| \phi' \|^{1+1/p'}_{L^\infty(\mathbb{R})} \| 1 \|_{L^{p'}(\{(t,x); 0 \le t+x \le \rho,\ 0 \le t-x \le 1\})} I \\
&= \rho^{-1/p} 2^{(2p'-1)/p'} l^2 \| \phi' \|^{1+1/p'}_{L^\infty(\mathbb{R})} I \\
&\le \rho^{-1} l I^p + p^{-p'/p} p'^{-1} 2^{2p'-1} l^{2p'-p'/p} \| \phi' \|^{p'+1}_{L^\infty(\mathbb{R})},
\end{aligned}$$

where $1/p' = 1 - 1/p$. By (4.2.1) and the estimates above, we have

$$f(\rho\delta) \le 2^{1/p'} m^2 \| u(t) \|_{L^1([0,T]; L^p([-1,1]))} + 2^{2p'-1} p^{-p'/p} p'^{-1} l^{2p'-p'/p} \| \phi' \|^{p'+1}_{L^\infty(\mathbb{R})}$$

and by taking $\rho \downarrow 0$, this is a contradiction to (4.1.5).

Appendix

Chapter A.1

Study of Semirelativistic System

A.1.1 Introduction

In this chapter, we study the following semirelativistic systems:

$$\begin{cases} i\partial_t u + (m_u^2 - \Delta)^{1/2} u = \lambda \overline{u} v, & t \in \mathbb{R},\ x \in \mathbb{R}, \\ i\partial_t v - (m_v^2 - \Delta)^{1/2} v = \frac{\overline{\lambda}}{c} u^2, & t \in \mathbb{R},\ x \in \mathbb{R}, \\ (u(0), v(0)) = (u_0, v_0), & x \in \mathbb{R}, \end{cases} \tag{A.1.1.1}$$

$$\begin{cases} i\partial_t u + (m_u^2 - \Delta)^{1/2} u = \lambda \overline{u} v, & t \in \mathbb{R},\ x \in \mathbb{R}, \\ i\partial_t v + (m_v^2 - \Delta)^{1/2} v = \frac{\overline{\lambda}}{c} u^2, & t \in \mathbb{R},\ x \in \mathbb{R}, \\ (u(0), v(0)) = (u_0, v_0), & x \in \mathbb{R}, \end{cases} \tag{A.1.1.2}$$

where $\lambda \in \mathbb{C} \backslash \{0\}$.

The aim of this chapter is to show that (A.1.1.1) has a similar property to (2.1.1) and (A.1.1.2) has a similar property to (3.1.1). We remark that the systems (A.1.1.1) and (A.1.1.2) are also regarded as a semirelativistic approximation of the Schrödinger system

$$\begin{cases} i\partial_t u + \frac{\sigma_1}{2m} \Delta u = \lambda \overline{u} v, \\ i\partial_t v + \frac{\sigma_2}{2M} \Delta v = \mu u^2, \end{cases} \tag{A.1.1.3}$$

where $\sigma_j \in \{-1, 1\}$. We refer the reader to [48, 49, 50, 51] for recent results on the Cauchy problem for (A.1.1.3). In the case of the Cauchy problem for (A.1.1.3) in the $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ setting, the signs of σ_1, σ_2 are not essential [50].

Since the charge of solutions to (A.1.1.1) is conserved, we have the following well-posedness of (A.1.1.1):

Theorem A.1.1.1. (A.1.1.1) *is time-globally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ setting with $s \geq 0$. Moreover, for $(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$, a pair of $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ solutions (u, v) corresponding to (u_0, v_0) satisfies*

$$\|u(t)\|_{L^2(\mathbb{R})} + c\|v(t)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})} + c\|v_0\|_{L^2(\mathbb{R})}.$$

We also have the following non-smoothness result for the solution map of (A.1.1.1).

Theorem A.1.1.2. *The solution map of* (A.1.1.1) *is not C^3 in the $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ setting with $-1/2 < s < 0$. In particular, if $-1/2 < s < 0$, then for some $t > 0$ and initial data*

$$u_0 = \mathfrak{F}^{-1}[\chi_{[1,\infty)} \langle \cdot \rangle^{-1/2-s-\varepsilon}] \in H^s(\mathbb{R})$$

and

$$v_0 = \mathfrak{F}^{-1}[\chi_{(-\infty,-1]}\langle\cdot\rangle^{-1/2-s-\varepsilon}] \in H^s(\mathbb{R})$$

with $0 < \varepsilon < -2s/3$,

$$\int_0^t U_1(t-t')\Big(U_2(-t')v_0 \int_0^{t'} U_1(t''-t')\Big(U_1(t'')u_0\overline{U_2(-t'')v_0}\Big)dt''\Big)\ dt' \notin H^s(\mathbb{R}), \tag{A.1.1.4}$$

where $U_1(t) = \exp(it(m_u^2 - \Delta)^{1/2})$ *and* $U_2(t) = \exp(it(m_v^2 - \Delta)^{1/2})$.

Theorem A.1.1.3. *The solution map of* (A.1.1.1) *is not* C^2 *in the* $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ *setting with* $s < -1/2$. *In particular, if* $s < -1/2$, *then for some* $t > 0$ *and a sequence of initial datum* $u_{0,k}$ *defined by*

$$v_{0,k} = u_{0,k} = k^{-s}\mathfrak{F}^{-1}[\chi_{[-1,1]}(\cdot - k) + \chi_{[-1,1]}(\cdot + k)]$$

then there exists $C > 0$ *such that for any* k, $\|u_{0,k}\|_{H^s} \le C$ *and*

$$\limsup_{k\to\infty} \left\| \int_0^t U_1(t-t')\Big(U_2(-t')v_{0,k}\overline{U_1(t')u_{0,k}}\Big)dt' \right\|_{H^s(\mathbb{R})} = \infty. \tag{A.1.1.5}$$

On the other hand, since the energy of solutions to (A.1.1.2) is conserved, we have the following well-posedness of (A.1.1.2):

Theorem A.1.1.4. *Let* $\lambda \in \mathbb{C}$. *For any* $u_0 \times v_0 \in H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$, *there exists a global solution to* (A.1.1.2). *Moreover, let* $u_{0,n}, v_{0,n}$ *and* $u_0, v_0 \in H^{1/2}(\mathbb{R})$ *satisfy* $(u_{0,n}, v_{0,n}) \to (u_0, v_0)$ *in* $H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ *as* $n \to \infty$, *and let* (u_n, v_n) *and* (u, v) *be the pairs of solutions of* (A.1.1.2) *with data* $(u_{0,n}, v_{0,n})$ *and* (u_0, v_0), *respectively. Then* $(u_n, v_n) \to (u, v)$ *in* $L^\infty(-T, T; H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R}))$ *for any* $T > 0$ *as* $n \to \infty$.

We also have the following non-smoothness result for the solution map of (A.1.1.2).

Theorem A.1.1.5. *The solution maps of* (A.1.1.2) *is not* C^2 *in the* $H^s(\mathbb{R})$ *setting with* $-1/2 < s < 1/2$. *In particular, if* $-1/2 < s < 1/2$, *then for some* $t > 0$, *initial data*

$$u_0 = \mathfrak{F}^{-1}[\chi_{[0,\infty)}\langle\cdot\rangle^{-1/2-s-\varepsilon}] \in H^s(\mathbb{R})$$

and

$$v_0 = \mathfrak{F}^{-1}[\chi_{(-\infty,0]}\langle\cdot\rangle^{-1/2-s-\varepsilon}] \in H^s(\mathbb{R})$$

with $0 < \varepsilon < 1/6 - s/3$,

$$\int_0^t U_1(t-t')\big(U_2(t')v_0\overline{U_1(t')u_0}\big)dt' \notin H^s(\mathbb{R}). \tag{A.1.1.6}$$

Theorem A.1.1.6. *The solution map of* (A.1.1.2) *is not* C^2 *in the* $H^s(\mathbb{R})$ *setting with* $s < -1/2$. *In particular, if* $s < -1/2$, *then for some* $t > 0$ *and a sequence of initial datum* $u_{0,k}$ *defined by*

$$u_{0,k} = v_{0,k} = k^{-s}\mathfrak{F}^{-1}[\chi_{[-1,1]}(\cdot - k)]$$

then there exists $C > 0$ *such that for any* k, $\|u_{0,k}\|_{H^s(\mathbb{R})} \le C$ *and*

$$\limsup_{k\to\infty} \left\| \int_0^t U_1(t-t')\overline{U_1(t')u_{0,k}}U_2(t')v_{0,k}dt' \right\|_{H^s(\mathbb{R})} = \infty \tag{A.1.1.7}$$

for some $t > 0$.

Theorems A.1.1.2, A.1.1.3, A.1.1.5, and A.1.1.6 can be proved by the same way as Theorems 2.1.2, 2.1.3, 2.1.4, and 2.1.5. It is because, there is no difference between U_1 and U_2 to show (A.1.1.4), (A.1.1.5), (A.1.1.6), and (A.1.1.7).

Theorem A.1.1.1 is obtained almost similarly to Theorem 2.1.1, since v in (A.1.1.1) can be regarded as $\overline{u}$. The difference between Theorems 2.1.1 and A.1.1.1 is the charge conservation. Since the charge of solutions to (A.1.1.1) is conserved, by the persistence of regularity, we have a unique pair of time-global solutions to (A.1.1.1). For details see Section A.1.2.

Also theorem A.1.1.2 is obtained almost similarly to Theorem 3.1.1, since the $H^{1/2}(\mathbb{R})$ norms of u and v can be controlled their conserved energy and charge. In Section A.1.3, we show only how the $H^{1/2}(\mathbb{R})$ norms of u and v are controlled.

A.1.2 Sketch of Proof of Theorem A.1.1.1

In this section, we show the charge conservation of solutions to (A.1.1.1) only. The construction of solutions and persistence of regularity can be obtained by a similar argument in the proof of Theorem 2.1.1 based on the following Banach space

$$\mathscr{X}^{s,b}[T_0, T_0+T] = X_-^{s,b}[T_0, T_0+T] \times X_+^{s,b}[T_0, T_0+T].$$

In particular, we show the following charge conservation law:

$$\|u(t)\|_{L^2(\mathbb{R})}^2 + c\|v(t)\|_{L^2(\mathbb{R})}^2 = \|u_0\|_{L^2(\mathbb{R})}^2 + c\|v_0\|_{L^2(\mathbb{R})}^2.$$

Although we can justify a formal proof of the $L^2(\mathbb{R})$ conservation by the approximation argument by smooth solutions, here, we derive the conservation laws directly without approximation. In particular, we derive the conservation law by using associated integral equations in the framework of Bourgain method as we studied in the previous sections. For the Schrödinger equation, there is a direct proof of the conservation laws in the framework of the Strichartz estimate [79]. To our knowledge, the direct proof of conservation law without smooth approximation had not been studied unless the Strichartz estimate hold. If one calculate the energy by integral equations without Strichartz estimate, a difficulty arises when one try to justify the each step of calculation. Especially, the integrability of each terms is a typical problem here, since only the boundedness of the Fourier restriction norms of solutions is available. To guarantee the integrability in each step, we use the following Lemma and Proposition.

Lemma A.1.2.1. *Let p and α satisfy $p \geq 1$ and $0 \leq \alpha \leq 1/p$. Let β, γ, κ satisfy $0 \leq \beta, \gamma, \kappa \leq 1/2$ and $\alpha + \beta + \gamma + \kappa = 1/p + 1/2 + \varepsilon$ with $\varepsilon > 0$. Then there exists a positive constant C such that the inequality*

$$\begin{aligned}&\|\langle \cdot + \delta_1\rangle^{-\alpha} f * g * h\|_{L^p(\mathbb{R})} \\ &\leq C\|\langle \cdot + \delta_2\rangle^{\beta} f\|_{L^2(\mathbb{R})}\|\langle \cdot + \delta_3\rangle^{\gamma} g\|_{L^2(\mathbb{R})}\|\langle \cdot + \delta_4\rangle^{\kappa} h\|_{L^2(\mathbb{R})}\end{aligned}$$

holds for any real numbers $\delta_1, \delta_2, \delta_3, \delta_4$ and any f, g, h such that all the norms on the right hand side are finite.

proof. By the Hölder and the Young inequalities,

$$\begin{aligned}&\|\langle \cdot + \delta_1\rangle^{-\alpha} f * g * h\|_{L^p(\mathbb{R})} \\ &\lesssim \|f * g * h\|_{L^{p_1}(\mathbb{R})} \\ &\lesssim \|f\|_{L^{p_2}(\mathbb{R})}\|g * h\|_{L^{p_3}(\mathbb{R})} \\ &\lesssim \|f\|_{L^{p_2}(\mathbb{R})}\|g\|_{L^{p_4}(\mathbb{R})}\|h\|_{L^{p_5}(\mathbb{R})} \\ &\lesssim \|\langle \cdot + \delta_2\rangle^{\beta} f\|_{L^2(\mathbb{R})}\|\langle \cdot + \delta_3\rangle^{\gamma} g\|_{L^2(\mathbb{R})}\|\langle \cdot + \delta_4\rangle^{\kappa} h\|_{L^2(\mathbb{R})},\end{aligned}$$

where

$$\frac{1}{p_1}=\frac{1}{p}-\alpha+\frac{\alpha\varepsilon}{\alpha+\beta+\gamma+\kappa}, \qquad \frac{1}{p_2}=\frac{1}{2}+\beta-\frac{\beta\varepsilon}{\alpha+\beta+\gamma+\kappa},$$
$$\frac{1}{p_3}=\frac{1}{p_1}+1-\frac{1}{p_2}, \qquad \frac{1}{p_4}=\frac{1}{2}+\gamma-\frac{\gamma\varepsilon}{\alpha+\beta+\gamma+\kappa},$$
$$\frac{1}{p_5}=\frac{1}{2}+\kappa-\frac{\kappa\varepsilon}{\alpha+\beta+\gamma+\kappa}.$$

Therefore, we obtain the lemma. Q.E.D.

Proposition A.1.2.1.

$$\begin{aligned}&\|\langle\tau\rangle^{-1}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi)\|_{L^1(\mathbb{R}_\tau\times\mathbb{R}_\xi\times\mathbb{R}_\sigma\times\mathbb{R}_\rho\times\mathbb{R}_\eta)}\\&\lesssim\|u\|_{X_\pm^{0,1/2}}\|v\|_{X_\pm^{0,1/2}}\|w\|_{X_\pm^{0,1/2}}\end{aligned}$$

for any $u,v,w\in X_\pm^{0,1/2}$.

proof. Let

$$N(\tau,\xi,\sigma,\rho,\varepsilon)=\max(|\tau|,\big|\sigma-\rho\pm|\xi-\eta|\big|,\big|\rho\pm|\eta|\big|,\big|\tau-\sigma\pm|\xi|\big|),$$

Then we have $|\xi|+|\xi-\eta|+|\eta|\le 4N$. We also separate the integral region as follows

$$\begin{aligned}B_1&=\{(\tau,\sigma,\xi,\rho,\eta)\ |N(\tau,\xi,\sigma,\rho,\varepsilon)=|\tau|\},\\B_2&=\{(\tau,\sigma,\xi,\rho,\eta)\ |N(\tau,\xi,\sigma,\rho,\varepsilon)=\big|\sigma-\rho\pm|\xi-\eta|\big|\},\\B_3&=\{(\tau,\sigma,\xi,\rho,\eta)\ |N(\tau,\xi,\sigma,\rho,\varepsilon)=\big|\rho\pm|\eta|\big|\},\\B_4&=\{(\tau,\sigma,\xi,\rho,\eta)\ |N(\tau,\xi,\sigma,\rho,\varepsilon)=\big|\tau-\sigma\pm|\xi|\big|\}.\end{aligned}$$

By Lemmas 2.2.7, A.1.2.1 and the Hölder inequality,

$$\begin{aligned}&\|\chi_{B_1}(\tau,\xi,\sigma,\eta)\langle\tau\rangle^{-1}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi)\|_{L^1(\mathbb{R}_\tau\times\mathbb{R}_\xi\times\mathbb{R}_\sigma\times\mathbb{R}_\rho\times\mathbb{R}_\eta)}\\&\lesssim\|\langle\tau\rangle^{-1/2}\langle\xi\rangle^{-1/4}\langle\eta\rangle^{-1/4}\\&\cdot\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi)\|_{L^1(\mathbb{R}_\tau\times\mathbb{R}_\xi\times\mathbb{R}_\sigma\times\mathbb{R}_\rho\times\mathbb{R}_\eta)}\\&\lesssim\big\|\langle\xi\rangle^{-1/4}\langle\eta\rangle^{-1/4}\|\langle\tau\pm|\xi-\eta|\rangle^{1/2}\widetilde{u}(\tau,\xi-\eta)\|_{L^2(\mathbb{R}_\tau)}\\&\cdot\|\langle\tau\pm|\eta|\rangle^{1/2}\widetilde{v}(\tau,\eta)\|_{L^2(\mathbb{R}_\tau)}\big\|_{L^2(\mathbb{R}_\xi;L^1(\mathbb{R}_\eta))}\|\langle\tau\pm|\xi|\rangle^{1/2}\widetilde{w}(\tau,\xi)\|_{L^2(\mathbb{R}_\tau\times\mathbb{R}_\xi)}\\&\lesssim\|u\|_{X_\pm^{0,1/2}}\|v\|_{X_\pm^{0,1/2}}\|w\|_{X_\pm^{0,1/2}}.\end{aligned}$$

Moreover,

$$\begin{aligned}&\|\chi_{B_2}(\tau,\xi,\sigma,\eta)\langle\tau\rangle^{-1}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi)\|_{L^1(\mathbb{R}_\tau\times\mathbb{R}_\xi\times\mathbb{R}_\sigma\times\mathbb{R}_\rho\times\mathbb{R}_\eta)}\\&\lesssim\|\langle\tau\rangle^{-1}\langle\xi\rangle^{-1/4}\langle\eta\rangle^{-1/4}\langle\sigma-\rho\pm|\xi-\eta|\rangle^{1/2}\\&\cdot\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi)\|_{L^1(\mathbb{R}_\tau\times\mathbb{R}_\xi\times\mathbb{R}_\sigma\times\mathbb{R}_\rho\times\mathbb{R}_\eta)}\\&\lesssim\big\|\langle\xi\rangle^{-1/4}\langle\eta\rangle^{-1/4}\|\langle\tau\pm|\xi-\eta|\rangle^{1/2}\widetilde{u}(\tau,\xi-\eta)\|_{L^2(\mathbb{R}_\tau)}\\&\cdot\|\langle\tau\pm|\eta|\rangle^{1/2}\widetilde{v}(\tau,\eta)\|_{L^2(\mathbb{R}_\tau)}\big\|_{L^2(\mathbb{R}_\xi;L^1(\mathbb{R}_\eta))}\|\langle\tau\pm|\xi|\rangle^{1/2}\widetilde{w}(\tau,\xi)\|_{L^2(\mathbb{R}_\tau\times\mathbb{R}_\xi)}\\&\lesssim\|u\|_{X_\pm^{0,1/2}}\|v\|_{X_\pm^{0,1/2}}\|w\|_{X_\pm^{0,1/2}}.\end{aligned}$$

The other integrals are estimated similarly. Q.E.D.

Then we show the charge conservation with Proposition A.1.2.1. Let $(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and let $T > 0$ sufficiently small. Then we have a pair of extensions $(u, v) \in X_-^{0,1/2} \times X_+^{0,1/2}$ of the solutions for the Cauchy problem (A.1.1.1) such that for any $t \in [0, T]$,

$$u(t) = U_1(t)u_0 - i\lambda \int_0^t U_1(t-t')\overline{u(t')}v(t')dt',$$
$$v(t) = U_2(-t)v_0 - ic^{-1}\overline{\lambda} \int_0^t U_2(t'-t)u(t')^2 dt'.$$

Then

$$\begin{aligned}&\|u(t)\|_{L^2(\mathbb{R})}^2 = \|U_1(t)u\|_{L^2(\mathbb{R})}^2\\ &= \left\| u_0 - i\lambda \int_0^t U_1(-t')\overline{u(t')}v(t')dt' \right\|_{L^2(\mathbb{R})}^2\\ &= \|u_0\|_{L^2(\mathbb{R})}^2 - 2\mathrm{Im}\left(\hat{u}_0 \middle| \lambda \int_0^t \mathfrak{F}[U_1(-t')\overline{u(t')}v(t')]dt' \right)\\ &+ \left\| \lambda \int_0^t \mathfrak{F}[U_1(-t')\overline{u}(t')v(t')]dt' \right\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

We have

$$\int_0^t f(t')dt' = \int \frac{\exp(it\tau) - 1}{i\tau} \hat{f}(\tau)d\tau$$

for any $f \in L^1(\mathbb{R})$ such that $\hat{f} \in \langle \cdot \rangle L^1(\mathbb{R})$. Moreover, the inequalities

$$\|\mathfrak{F}[\overline{u}v]\|_{L^\infty(\mathbb{R}_\xi; L^1(\mathbb{R}_t))} \leq \|u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{X_-^{0,1/2}}\|v\|_{X_+^{0.1/2}}$$

hold by the Hölder inequality and

$$\iiiint_{\mathbb{R}^4} \frac{\exp(it\cdot) - 1}{i\cdot} \overline{\widetilde{u}(\rho-\sigma, \eta-\xi)}\widetilde{v}(\rho, \eta)\overline{\widetilde{u}(\sigma - \cdot, \xi)}\, d\xi\, d\sigma\, d\eta\, d\rho \in L^1(\mathbb{R})$$

by Proposition A.1.2.1. Then

$$\begin{aligned}&\left\| \lambda \int_0^t \mathfrak{F}[U_1(-t')\overline{u}(t')v(t')]dt' \right\|_{L^2(\mathbb{R})}^2\\ &= 2\mathrm{Re} \int_{\mathbb{R}} \int_0^t \lambda \mathfrak{F}[\overline{u(t')}v(t')]\, \overline{\lambda \mathfrak{F}\left[\int_0^{t'} U_1(t'-t'')\overline{u(t'')}v(t'')dt'' \right]}\, dt' d\xi\\ &= -2\mathrm{Im} \int_{\mathbb{R}} \int_0^t \lambda \mathfrak{F}[\overline{u(t')}v(t')]\overline{(\mathfrak{F}[U_1(t')u_0] - \mathfrak{F}[u(t')])}dt' d\xi\\ &= 2\mathrm{Im}\left(\hat{u}_0 \middle| \lambda \int_0^t \mathfrak{F}[U_1(t')\overline{u(t')}v(t')]dt' \right)\\ &+ 2\mathrm{Im}\ \lambda \iiiiint_{\mathbb{R}^5} \frac{\exp(it\tau) - 1}{i\tau} \overline{\widetilde{u}(\rho-\sigma, \eta-\xi)}\widetilde{v}(\rho, \eta)\overline{\widetilde{u}(\sigma-\tau, \xi)}\, d\tau\, d\xi\, d\sigma\, d\eta\, d\rho.\end{aligned}$$

Finally we obtain

$$\begin{aligned}&\|u(t)\|_{L^2(\mathbb{R})}^2 - \|u_0\|_{L^2(\mathbb{R})}^2\\ &= 2\mathrm{Im}\ \lambda \iiiiint_{\mathbb{R}^5} \frac{\exp(it\tau) - 1}{i\tau} \overline{\widetilde{u}(\rho-\sigma, \eta-\xi)\widetilde{u}(\sigma-\tau, \xi)}\widetilde{v}(\rho, \eta)\, d\tau\, d\xi\, d\sigma\, d\eta\, d\rho.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|v(t)\|_{L^2(\mathbb{R})}^2 &= \|U_2(-t)v\|_{L^2(\mathbb{R})}^2\\ &= \|v_0\|_{L^2(\mathbb{R})} - 2\mathrm{Im}\left(\hat{v}_0\middle|c^{-1}\overline{\lambda}\int_0^t \mathfrak{F}\left[U_2(t')u(t')^2\right]dt'\right)\\ &+ \left\|c^{-1}\overline{\lambda}\int_0^t \mathfrak{F}[U_2(-t')u(t')^2dt']\right\|_{L^2(\mathbb{R})}^2\end{aligned}$$

and

$$\begin{aligned}&\left\|c^{-1}\overline{\lambda}\int_0^t \mathfrak{F}_x[U_2(t')u(t')^2]dt'\right\|_{L^2(\mathbb{R})}^2\\ &= -2\mathrm{Im}\int_{\mathbb{R}}\int_0^t c^{-1}\overline{\lambda}\mathfrak{F}[u(t')^2]\,\overline{\mathfrak{F}\left[ic^{-1}\overline{\lambda}\int_0^{t'} U_2(t''-t')u(t'')^2dt''\right]}dt'd\xi\\ &= 2\mathrm{Im}\left(\hat{v}_0\middle|c^{-1}\overline{\lambda}\int_0^t \mathfrak{F}[U_2(t')u(t')^2]dt'\right)\\ &+ \frac{2}{c}\mathrm{Im}\ \overline{\lambda}\iiiint\!\!\int_{\mathbb{R}^5}\frac{\exp(it\tau)-1}{i\tau}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)\overline{\widetilde{v}(\sigma-\tau,\xi)}\,d\tau\,d\xi\,d\sigma\,d\eta\,d\rho.\end{aligned}$$

Then

$$\begin{aligned}&\|v(t)\|_{L^2(\mathbb{R})}^2 - \|v_0\|_{L^2(\mathbb{R})}^2\\ &= 2c^{-1}\mathrm{Im}\ \overline{\lambda}\iiiint\!\!\int_{\mathbb{R}^5}\frac{\exp(it\tau)-1}{i\tau}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)\overline{\widetilde{v}(\sigma-\tau,\xi)}\,d\tau\,d\xi\,d\sigma\,d\eta\,d\rho.\end{aligned}$$

In addition,

$$\begin{aligned}&-\mathrm{Im}\overline{\lambda}\iiiint\!\!\int_{\mathbb{R}^5}\frac{\exp(it\tau)-1}{i\tau}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)\overline{\widetilde{v}(\sigma-\tau,\xi)}\,d\tau\,d\xi\,d\sigma\,d\eta\,d\rho\\ &= \mathrm{Im}\lambda\iiiint\!\!\int_{\mathbb{R}^5}\frac{\exp(-it\tau)-1}{i(-\tau)}\overline{\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)}\ \widetilde{v}(\sigma-\tau,\xi)\,d\tau\,d\xi\,d\sigma\,d\eta\,d\rho\\ &= \mathrm{Im}\lambda\iiiint\!\!\int_{\mathbb{R}^5}\frac{\exp(-it\tau)-1}{i(-\tau)}\overline{\widetilde{u}(\tau+\rho'-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)}\ \widetilde{v}(\rho',\xi)\,d\xi\,d\sigma\,d\eta\,d\rho'\,d\tau\\ &= \mathrm{Im}\lambda\iiiint\!\!\int_{\mathbb{R}^5}\frac{\exp(it\tau')-1}{i\tau'}\overline{\widetilde{u}(\rho'-\sigma',\xi-\eta)\widetilde{u}(\sigma'-\tau',\eta)}\ \widetilde{v}(\rho',\xi)\,d\xi\,d\sigma'\,d\eta\,d\rho'\,d\tau'\\ &= \mathrm{Im}\lambda\iiiint\!\!\int_{\mathbb{R}^5}\frac{\exp(it\tau')-1}{i\tau'}\overline{\widetilde{u}(\rho'-\sigma',\eta'-\xi')\widetilde{u}(\sigma'-\tau',\xi')}\ \widetilde{v}(\rho',\eta')\,d\xi'\,d\sigma'\,d\eta'\,d\rho'\,d\tau',\end{aligned}$$

where $\rho'=\sigma-\tau$, $\sigma'=\rho-\tau$, $\tau'=-\tau$, $\xi'=\eta$, and $\eta'=\xi$. Finally, we have

$$\|u(t)\|_{L^2(\mathbb{R})}^2 + c\|v(t)\|_{L^2(\mathbb{R})}^2 = \|u_0\|_{L^2(\mathbb{R})}^2 + c\|v_0\|_{L^2(\mathbb{R})}^2$$

for $t\in[0,T]$.

A.1.3 Proof of Theorem A.1.1.2

Here, we show how the $H^{1/2}(\mathbb{R})$ norms of u and v are controlled. At first, we show the conservation law of charge and energy.

Lemma A.1.3.1. *Let* $(u_0, v_0) \in H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$. *Let* $(u, v) \in C(\mathbb{R}; H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})) \cap C^1(\mathbb{R}; H^{-1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R}))$ *be solutions to the integral equations* (A.1.1.2) *for the initial data* (u_0, v_0). *Then* $(Q(u(t), v(t)) = (Q(u_0, v_0))$ *for any* t, *where*

$$Q(f, g) = \|f\|^2_{L^2(\mathbb{R})} + c\|g\|^2_{L^2(\mathbb{R})}.$$

proof. The following formal calculations are justified by extending $L^2(\mathbb{R})$ scalar product to $H^{-1/2}(\mathbb{R})$-$H^{1/2}(\mathbb{R})$ duality:

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R})} &= 2\mathrm{Re}\langle \partial_t u(t) \mid u(t)\rangle_{H^{1/2}(\mathbb{R})} \\
&= 2\mathrm{Im}\langle i\partial_t u(t) \mid u(t)\rangle_{H^{1/2}(\mathbb{R})} \\
&= 2\mathrm{Im}\langle -(m_u^2 - \Delta)^{1/2}\ u(t) + \lambda\overline{u(t)}v(t) \mid u(t)\rangle_{H^{1/2}(\mathbb{R})} \\
&= 2\mathrm{Im}(\lambda v(t) \mid u(t)^2), \\
\frac{d}{dt}\|v(t)\|^2_{L^2(\mathbb{R})} &= 2\mathrm{Re}\langle \partial_t v(t) \mid v(t)\rangle_{H^{1/2}(\mathbb{R})} \\
&= 2\mathrm{Im}\langle i\partial_t v(t) \mid v(t)\rangle_{H^{1/2}(\mathbb{R})} \\
&= 2\mathrm{Im}\langle -(m_v^2 - \Delta)^{1/2}\ v(t) + c^{-1}\overline{\lambda}u(t)^2 \mid v(t)\rangle_{H^{1/2}(\mathbb{R})} \\
&= -\frac{2}{c}\mathrm{Im}(\lambda v(t) \mid u(t)^2).
\end{aligned}$$

Therefore, we obtain that

$$\|u(t)\|^2_{L^2(\mathbb{R})} + c\|v(t)\|^2_{L^2(\mathbb{R})} = \|u_0\|^2_{L^2(\mathbb{R})} + c\|v_0\|^2_{L^2(\mathbb{R})}$$

for any t.

Q.E.D.

Lemma A.1.3.2. *Let* $(u_0, v_0) \in H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$. *Let* $(u, v) \in C(\mathbb{R}; H^1(\mathbb{R}) \times H^1(\mathbb{R})) \cap C^1(\mathbb{R}; L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ *be solutions to the integral equations* (A.1.1.2) *for the initial data* (u_0, v_0). *Then* $E(u(t), v(t)) = E(u_0, v_0)$ *for any* t, *where*

$$E(f, g) = \|(m_u^2 - \Delta)^{\frac{1}{4}} f\|^2_{L^2(\mathbb{R})} + \frac{c}{2}\|(m_v^2 - \Delta)^{1/4} g\|^2_{L^2(\mathbb{R})} - \mathrm{Re}(\lambda g \mid f^2) \tag{A.1.3.1}$$

for any t.

proof. $\|(m_u^2 - \Delta)^{1/4} u\|^2_{L^2(\mathbb{R})}$ and $\|(m_v^2 - \Delta)^{1/4} v\|^2_{L^2(\mathbb{R})}$ are differentiable and we have

$$\begin{aligned}
\frac{d}{dt}\|(m_u^2 - \Delta)^{\frac{1}{4}} u(t)\|^2_{L^2(\mathbb{R})} &= 2\mathrm{Re}(-i\partial_t u(t) + \lambda\overline{u(t)}v(t) \mid \partial_t u(t)) \\
&= \mathrm{Re}(\lambda v(t) \mid \partial_t(u(t)^2)) \\
\frac{d}{dt}\|(m_v^2 - \Delta)^{\frac{1}{4}} v(t)\|^2_{L^2(\mathbb{R})} &= 2\mathrm{Re}\Big(- i\partial_t v + \frac{\overline{\lambda}}{c}u(t)^2 \Big| \partial_t v(t)\Big) \\
&= \frac{2}{c}\mathrm{Re}(\partial_t(\lambda v)(t) \mid u(t)^2).
\end{aligned}$$

Therefore, we obtain that $E(u(t), v(t)) = E(u_0, v_0)$ for any t, Q.E.D.

By Lemmas A.1.3.1 and A.1.3.2, we obtain the $H^{1/2}(\mathbb{R})$ boundedness of solutions to (A.1.1.2), as shown below. By using this $H^{1/2}(\mathbb{R})$ boundedness of solutions, Theorem A.1.1.2 follows from a similar proof to Theorem 3.1.1 with the corresponding approximation integral equations. At the last of this section, we show the $H^{1/2}(\mathbb{R})$ boundedness of solutions to (A.1.1.2).

Proposition A.1.3.1. *Let* $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ *and let* $(u, v) \in C([-T, T]; H^1(\mathbb{R}) \times H^1(\mathbb{R}))$ *be solutions to the integral equations* (A.1.1.2) *for the initial data* (u_0, v_0) *with some* $T > 0$. *Then*

$$\sup_{t\in[-T,T]} \|u(t)\|_{H^{1/2}(\mathbb{R})} + \|v(t)\|_{H^{1/2}(\mathbb{R})} \lesssim \sqrt{E(u_0, v_0)} + Q(u_0, v_0).$$

proof. It is enough to show

$$\begin{aligned}&\sup_{t\in[-T,T]} \|(m_u^2 - \Delta)^{1/2} u(t)\|_{L^2(\mathbb{R})}^2 + \frac{c}{2}\|(m_v^2 - \Delta)^{1/2} v(t)\|_{L^2(\mathbb{R})}^2 \\ &\lesssim E(u_0, v_0) + Q(u_0, v_0)^2.\end{aligned}$$

We note that by the Hölder and Gagliardo-Nirenberg inequalities,

$$\begin{aligned}|(\lambda v(t) \mid u^2(t))| &\lesssim \|v(t)\|_{L^2(\mathbb{R})} \|u(t)\|_{L^4(\mathbb{R})}^2 \\ &\lesssim \|v(t)\|_{L^2(\mathbb{R})} \|u(t)\|_{L^2(\mathbb{R})} \|u(t)\|_{\dot{H}^{1/2}(\mathbb{R})} \\ &\leq \frac{1}{\sqrt{c}} Q(u_0, v_0) \|(m_u^2 - \Delta)^{1/2} u(t)\|_{L^2(\mathbb{R})}.\end{aligned}$$

Then

$$\begin{aligned}&\|(m_u^2 - \Delta)^{\frac{1}{4}} u(t)\|_{L^2(\mathbb{R})}^2 + \frac{c}{2}\|(m_v^2 - \Delta)^{1/4} v(t)\|_{L^2(\mathbb{R})}^2 \\ &= E(u_0, v_0) + \mathrm{Re}(\lambda v(t) \mid u(t)^2) \\ &\leq E(u_0, v_0) + \frac{1}{\sqrt{c}} Q(u_0, v_0) \|(m_u^2 - \Delta)^{1/2} u(t)\|_{L^2(\mathbb{R})}.\end{aligned}$$

This shows

$$\begin{aligned}&\|(m_u^2 - \Delta)^{\frac{1}{4}} u(t)\|_{L^2(\mathbb{R})}^2 + \frac{c}{2}\|(m_v^2 - \Delta)^{1/4} v(t)\|_{L^2(\mathbb{R})}^2 \\ &\leq 2E(u_0, v_0) + \frac{4}{c} Q(u_0, v_0)^2.\end{aligned}$$

Q.E.D.

Chapter A.2

Study of Weighted Integral

A.2.1 Introduction

In this chapter, we revisit Lemma 1.3.11 by studying the boundedness of integral operators of convolution type in the Lebesgue space with weights. Moreover, a special attention will be made on an optimality criterion with respect to the growth rate of weights.

To illustrate the problem, we revisit the standard property that the Sobolev space $H^s(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^2(\mathbb{R}^n)$ forms an algebra for $s > n/2$ from the point of view from the weighted $L^2(\mathbb{R}^n)$-boundedness of convolution. The corresponding bilinear estimate in the Sobolev space takes the form

$$\|uv\|_{H^s(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}. \tag{A.2.1.1}$$

The bilinear estimate of this type was may be traced back at least to the paper by Saut and Temam [85]. There are many papers on further refinements and improvements on this subject as well as various applications to nonlinear partial differential equations. (see for instance [2, 23, 28, 31, 34, 35, 56, 57, 60, 61, 69, 73, 83, 85, 87, 89, 90] and references therein.)

In the Fourier representation, multiplication of functions is realized by convolution of the corresponding Fourier transformed functions:

$$\mathfrak{F}(uv)(\xi) = (2\pi)^{n/2}(\hat{u} * \hat{v})(\xi) = (2\pi)^{n/2}\int_{\mathbb{R}^n} \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta$$

and the estimate (A.2.1.1) is equivalent to the bilinear estimate of the form

$$\|\omega(\hat{u} * \hat{v})\|_{L^2(\mathbb{R}^n)} \leq C\|\omega\hat{u}\|_{L^2(\mathbb{R}^n)}\|\omega\hat{v}\|_{L^2(\mathbb{R}^n)}$$

with $\omega(\xi) = (1+|\xi|^2)^{s/2}$, which is also rewritten as

$$\left\|\omega\Big(\Big(\frac{\hat{u}}{\omega}\Big) * \Big(\frac{\hat{v}}{\omega}\Big)\Big)\right\|_{L^2(\mathbb{R}^n)} \leq C\|\hat{u}\|_{L^2(\mathbb{R}^n)}\|\hat{v}\|_{L^2(\mathbb{R}^n)}. \tag{A.2.1.2}$$

By a duality argument, (A.2.1.2) is equivalent to the trilinear estimate of the form

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \omega(\xi)\frac{1}{\omega(\xi-\eta)}\frac{1}{\omega(\eta)}\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(\xi)\,d\eta\,d\xi\right| \\ &\leq C\|\hat{u}\|_{L^2(\mathbb{R}^n)}\|\hat{v}\|_{L^2(\mathbb{R}^n)}\|\hat{w}\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{A.2.1.3}$$

By a simple change of variables, (A.2.1.3) is equivalent to

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \omega(\xi+\eta) \frac{1}{\omega(\xi)} \frac{1}{\omega(\eta)} \hat{u}(\xi)\hat{v}(\eta)\hat{w}(\xi+\eta)\, d\eta\, d\xi \right| \\ \leq C\|\hat{u}\|_{L^2(\mathbb{R}^n)}\|\hat{v}\|_{L^2(\mathbb{R}^n)}\|\hat{w}\|_{L^2(\mathbb{R}^n)}. \tag{A.2.1.4}$$

This gives a motivation to study the boundedness of the integrals of the form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w_0(x+y)w_1(x)w_2(y)f(x+y)g(x)h(y)\, dx\, dy \tag{A.2.1.5}$$

with weight functions w_0, w_1, w_2, where w_1 and w_2 are supposedly the inverse weight of w_0.

The following theorem is basic in this direction.

Theorem A.2.1.1. *Let $2 \leq p \leq \infty$ and let w_0, w_1, w_2 be non-negative, continuous functions on $[0,\infty)$ satisfying*

$$M_1 \equiv \sup_{r>0} w_0^{\#}(2r)w_2(r)\|w_1(|\cdot|)\|_{L^p(B(r))} < \infty, \tag{A.2.1.6}$$

$$M_2 \equiv \sup_{r>0} w_0^{\#}(2r)w_1(r)\|w_2(|\cdot|)\|_{L^p(B(r))} < \infty, \tag{A.2.1.7}$$

where

$$w_0^{\#}(r) = \sup_{0\leq\rho\leq r} w_0(\rho),$$
$$B(r) = \{x \in \mathbb{R}^n; |x| \leq r\}.$$

Then, the trilinear estimate

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w_0(|x+y|)\ w_1(|x|)\ w_2(|y|)\ |f(x+y)g(x)h(y)|\, dx\, dy \\ \leq (M_1+M_2)\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)}\|h\|_{L^{p'}(\mathbb{R}^n)} \tag{A.2.1.8}$$

holds for all $f \in L^p(\mathbb{R}^n)$, g, $h \in L^{p'}(\mathbb{R}^n)$, where p' is the dual exponent defined by $1/p + 1/p' = 1$.

proof. For $f \in L^p(\mathbb{R}^n)$ we define the translation by $y \in \mathbb{R}^n$ by $(\tau_y f)(x) = f(x+y)$. For $S \subset \mathbb{R}^n$, we denote by χ_S its characteristic function. Then, by the Hölder and Minkowski inequalities, we obtain

$$\begin{aligned}
&\iint_{|x|\leq|y|} w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\, dx\, dy \\
&\leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} w_0^{\#}(2|y|)\chi_{B(|y|)}(x)w_1(|x|)w_2(|y|)|\tau_y f(x)g(x)h(y)|\, dx\, dy \\
&\leq \int_{\mathbb{R}^n} w_0^{\#}(2|y|)\|\chi_{B(|y|)}w_1(|\cdot|)\|_{L^p(\mathbb{R}^n)}\|\tau_y f\cdot g\|_{L^{p'}(\mathbb{R}^n)}w_2(|y|)|h(y)|\, dy \\
&\leq M_1\|\|\tau_y f\cdot g\|_{L^{p'}(\mathbb{R}^n)}\|_{L^p(\mathbb{R}^n_y)}\|h\|_{L^{p'}(\mathbb{R}^n)} \\
&= M_1\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)}\|h\|_{L^{p'}(\mathbb{R}^n)},
\end{aligned}$$

where $L^p(\mathbb{R}^n_y)$ is the $L^p(\mathbb{R}^n)$ norm for the variable y. Similarly,

$$\begin{aligned}
&\iint_{|x|\geq|y|} w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\,dx\,dy \\
&\leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} w_0^{\#}(2|x|)\chi_{B(|x|)}(y)w_1(|x|)w_2(|y|)|\tau_x f(y)g(x)h(y)|\,dx\,dy \\
&\leq \int_{\mathbb{R}^n} w_0^{\#}(2|x|)\|\chi_{B(|x|)}w_2(|\cdot|)\|_{L^p(\mathbb{R}^n)}\|\tau_x f\cdot h\|_{L^{p'}(\mathbb{R}^n)}w_1(|x|)|g(x)|\,dx \\
&\leq M_2\|\|\tau_x f\cdot h\|_{L^{p'}(\mathbb{R}^n)}\|_{L^p(\mathbb{R}^n_x)}\|g\|_{L^{p'}(\mathbb{R}^n)} \\
&= M_2\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)}\|h\|_{L^{p'}(\mathbb{R}^n)}.
\end{aligned}$$

Summing those inequalities, we have (A.2.1.8). Q.E.D.

Corollary A.2.1.1. *Let $2\leq p\leq\infty$ and let w_0, w_1, w_2 be non-negative, continuous functions on $[0,\infty)$ satisfying*

$$M_1' = \sup_{r>0} w_0(2r)w_2(r)\|w_1(|\cdot|)\|_{L^p(B(r))} < \infty, \tag{A.2.1.9}$$

$$M_2' = \sup_{r>0} w_0(2r)w_1(r)\|w_2(|\cdot|)\|_{L^p(B(r))} < \infty, \tag{A.2.1.10}$$

and the estimate

$$w_0(r)\leq C'w_0(R) \tag{A.2.1.11}$$

for any r and R with $0\leq r\leq R$ with $C'\geq 1$ independent of r and R. Then, the trilinear estimate

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w_0(|x+y|)\; w_1(|x|)\; w_2(|y|)\; |f(y+x)g(x)h(y)|\,dx\,dy \\
&\leq C'(M_1'+M_2')\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)}\|h\|_{L^{p'}(\mathbb{R}^n)}
\end{aligned}$$

holds for all $f\in L^p(\mathbb{R}^n)$, g, $h\in L^{p'}(\mathbb{R}^n)$.

proof. By (A.2.1.11), we have $w_0^{\#}(2r)\leq C'w_0(2r)$ for any $r\geq 0$. Then, the corollary follows from Theorem A.2.1.1 Q.E.D.

The bilinear estimate (A.2.1.1) follows by choosing $p=2$, $w_0(r)=(1+r^2)^{s/2}$, $w_1(r)=w_2(r)=(1+r^2)^{-s/2}$ with $s>n/2$, which ensures the required square integrability. Moreover, Lemma 1.3.11 also follows by choosing $p=2$, $w_0(r)=(1+r^2)^{-a/2}$, $w_1(r)=(1+r^2)^{-b/2}$, and $w_2(r)=(1+r^2)^{-c/2}$ which also ensures the required square integrability. A natural question then arises in connection with minimal growth rate at infinity in space for w_0, $1/w_1$, $1/w_2$. Weight functions of the form $w(r)=(1+r^2)^{n/2}(1+\log(1+r))^s$ with $s>1/2$ may be the first candidate with $w_0=w$, $w_1=w_2=1/w$. This is not optimal since $w(r)=(1+r^2)^{n/2}(1+\log(1+r))^{1/2}\big(1+\log(1+\log(1+r))\big)^s$ with $s>1/2$ has a slower growth with keeping the required square integrability.

To describe emerging extra logarithmic factors in such an iteration procedure, it is convenient to introduce the following set $\mathscr{F}$ consisting of positive, continuous functions w on $[0,\infty)$ satisfying $1/w\in L^1_{\mathrm{loc}}(0,\infty)$ and the following assumptions (A1) and (A2):

(A1) For any $a\in\mathbb{R}$, there exists $C_a\geq 1$ such that for any r and R with $0\leq r\leq R$, w satisfies the inequality

$$w(r)\left(\int_0^r\frac{1}{w(\rho)}d\rho+1\right)^a\leq C_a w(R)\left(\int_0^R\frac{1}{w(\rho)}d\rho+1\right)^a.$$

(A2) There exists $C > 0$ such that the inequality

$$w(2r) \leq Cw(r)$$

holds for all $r > 0$.

Example 1. The function w defined by $w(r) = 1 + r$ belongs to $\mathscr{F}$ with $C_a = 1$ for $a \geq -1$, $C_a = e^{a+1}(-a)^{-a}$ for $a < -1$, and $C = 2$.

Example 2. The function w defined by $w(r) = (1 + r)^s$ with $s > 1$ belongs to $\mathscr{F}$ with $C_a = 1$ for $a \geq -s$,

$$C_a = (-a)^{-a}(a + s - as)^{\frac{as-a-s}{s-1}} s^{\frac{2s+a-as}{s-1}}$$

for $a < -s$, and $C = 2^s$.

Example 3. The function w defined by $w(r) = (1+r^2)^{s/2}$ for $s \geq 1$ belong to $\mathscr{F}$ with $C_a = 1$ for $a \geq 0$,

$$C_a = s^a(-a)^{-a} r_{s,a}^a (1 + r_{s,a}^2)^{-a+(a-1)s/2}$$

for $a < 0$, where $r_{s,a}$ is defined uniquely by

$$r_{s,a}(1 + r_{s,a}^2)^{s/2-1}\left(\int_0^{r_{s,a}} (1 + \rho^2)^{-s/2} d\rho + 1\right) = \frac{|a|}{s}$$

and $C = 2^s$.

Example 4. Let $w(r) = 1 + \log(1 + r)$ and $a = -2$. Then,

$$w(r)\left(\int_0^r \frac{1}{w(\rho)} d\rho + 1\right)^{-2} \leq w(r)\left(\int_0^r \frac{1}{1+\rho} d\rho + 1\right)^{-2} = \frac{1}{w(r)} \to 0$$

as $r \to \infty$. This means $w \notin \mathscr{F}$.

Example 5. The function w defined by $w(r) = (1 + r)(1 + \log(1 + r))$ belongs to $\mathscr{F}$ with $C_a = 1$ for $a \geq 0$,

$$C_a = (-a)^{-a}(1 + \tilde{r}_{s,a})^{-1}(1 + \log(1 + \tilde{r}_{s,a}))^{-1}(2 + \log(\tilde{r}_{s,a}))^a$$

for $a < 0$, where $\tilde{r}_{s,a}$ is uniquely defined by

$$(2 + \log(1 + \tilde{r}_{s,a}))\Big(1 + \log(1 + \log(1 + \tilde{r}_{s,a}))\Big) = |a|,$$

and $C = 2 + 2\log 2$.

Remark A.2.1.1. *For $w \in \mathscr{F}$, we apply (A1) with $a = 0$ to obtain*

$$\begin{aligned}
&\int_0^r \frac{1}{w(\rho)} d\rho \\
&\leq \int_0^{2r} \frac{1}{w(\rho)} d\rho = \int_0^r \frac{1}{w(\rho)} d\rho + \int_0^r \frac{1}{w(\rho + r)} d\rho \\
&\leq (1 + C_0) \int_0^r \frac{1}{w(\rho)} d\rho. \qquad \text{(A.2.1.12)}
\end{aligned}$$

Theorem A.2.1.2. *Let $2 \le p < \infty$ and let $w \in \mathscr{F}$. Let w_0, w_1, w_2 be defined by*

$$w_0(r) = (1+r)^{(n-1)/p} w(r)^{1/p} \Big(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \Big)^{-a},$$
$$w_1(r) = (1+r)^{-(n-1)/p} w(r)^{-1/p} \Big(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \Big)^{-b},$$
$$w_2(r) = (1+r)^{-(n-1)/p} w(r)^{-1/p} \Big(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \Big)^{-c}$$

with $a, b, c \in \mathbb{R}$ satisfying either (i) or (ii):

(i) $a+b+c \ge 1/p, \qquad a+b>0, \qquad a+c>0.$

(ii) $a+b+c > 1/p, \qquad a+b \ge 0, \qquad a+c \ge 0.$

Then, there exists $C>0$ such that the trilinear estimate

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} w_0(|x+y|) w_1(|x|) w_2(|y|) |f(x+y) g(x) h(y)| dx\, dy \le C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)} \|h\|_{L^{p'}(\mathbb{R}^n)} \tag{A.2.1.13}$$

holds for all $f \in L^p(\mathbb{R}^n)$, $g, h \in L^{p'}(\mathbb{R}^n)$.

Remark A.2.1.2. *In the case $\int_0^\infty w^{-1}(\rho) d\rho < \infty$, we can choose any a, b, c for* (A.2.1.13). *In the case where $p = 2$ and $b = c = 0$, assumption (i) is equivalent to $a \ge 1/2$. In the case where $p = 2$ and $b = c > 0$, assumption (i) is equivalent to $a \ge 1/2 - 2b$ with $a > -b$. In the case where $p = 2$ and $-a = b = c$, assumption (i) breaks down and (ii) is equivalent to $-a = b = c > 1/2$.*

Proof of Theorem A.2.1.2. We prove that w_0, w_1, w_2 defined in the theorem satisfy the assumptions (A.2.1.9)-(A.2.1.11) in Corollary A.2.1.1. Let r and R satisfy $0 \le r \le R$. By (A1),

$$w(r) \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^{-ap} \le C_{-ap} w(R) \left(\int_0^R \frac{1}{w(\rho)} d\rho + 1 \right)^{-ap},$$

which yields

$$w_0(r) \le C_{-ap}^{1/p} w_0(R). \tag{A.2.1.14}$$

By (A2) and (A.2.1.12),

$$\begin{aligned} w_0(2r) &= (1+2r)^{(n-1)/p} w(2r)^{1/p} \left(\int_0^{2r} \frac{1}{w(r)} d\rho + 1 \right)^{-a} \\ &\le 2^{(n-1)/p} (1+r)^{(n-1)/p} C^{1/p} w(r)^{1/p} (1+C_0)^{(-a)_+} \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^{-a} \\ &= 2^{(n-1)/p} C^{1/p} (1+C_0)^{(-a)_+} w_0(r), \end{aligned} \tag{A.2.1.15}$$

which yields

$$w_0(2r) w_1(r) \le 2^{(n-1)/p} C^{1/p} (1+C_0)^{(-a)_+} \left(\int_0^r \frac{1}{w(r)} d\rho + 1 \right)^{-a-b}. \tag{A.2.1.16}$$

We estimate $w_2(|\cdot|)$ in $L^p(B(r))$ as

$$\|w_2(|\cdot|)\|_{L^p(B(r))} \le \omega_{n-1}^{1/p}\left(\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma+1\right)^{-pc}d\rho\right)^{1/p}, \tag{A.2.1.17}$$

where ω_{n-1} is the surface measure of the unit ball. To estimate the right hand side of (A.2.1.17) and M_1' of Corollary A.2.1.1, we distinguish four cases:

(i) $c \le 0$. (ii) $0 < c < 1/p$. (iii) $c = 1/p$. (iv) $c > 1/p$.

(i) In the case where $c \le 0$, we estimate

$$\begin{aligned}\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma+1\right)^{-pc}d\rho &\le \int_0^r \frac{1}{w(\rho)}\left(\int_0^r \frac{1}{w(\sigma)}d\sigma+1\right)^{-pc}d\rho\\ &\le \left(\int_0^r \frac{1}{w(\rho)}d\rho+1\right)^{1-pc}.\end{aligned}$$

Then, M_1' is estimated as follows:

$$\begin{aligned}M_1' &\le \sup_{r>0} 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}\left(\int_0^r \frac{1}{w(r)}d\rho+1\right)^{1/p-a-b-c}\\ &= 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}.\end{aligned}$$

(ii) In the case where $0 < c < 1/p$, we estimate

$$\begin{aligned}\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma+1\right)^{-pc}d\rho &= \frac{1}{1-pc}\left(\left(\int_0^r \frac{1}{w(\sigma)}d\sigma+1\right)^{1-pc}-1\right)\\ &\le \frac{1}{1-pc}\left(\int_0^r \frac{1}{w(\sigma)}d\sigma+1\right)^{1-pc}.\end{aligned}$$

Then, M_1' is estimated as follows:

$$M_1' \le \frac{1}{1-pc}2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}.$$

(iii) In the case where $c = 1/p$, we estimate

$$\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma+1\right)^{-1}d\rho = \log\left(1+\int_0^r \frac{1}{w(\rho)}d\rho\right).$$

Since $a+b>0$, M_1' is estimated as follows:

$$\begin{aligned}M_1' &\le C^{1/p}(1+C_0)^{(-a)_+}\\ &\quad\cdot \sup_{r>0} 2^{(n-1)/p}\left(\int_0^r \frac{1}{w(r)}d\rho+1\right)^{-a-b}\log\left(1+\int_0^r \frac{1}{w(\rho)}d\rho\right)\\ &= 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}\sup_{r\ge 1} r^{-a-b}\log r\\ &= 2^{(n-1)/p}C^{1/p}(1+C_0)^{(-a)_+}\frac{1}{e(a+b)}.\end{aligned}$$

(iv) In the case where $c > 1/p$, we estimate

$$\begin{aligned}
&\int_0^r \frac{1}{w(\rho)}\left(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\right)^{-pc} d\rho \\
&= \frac{1}{1-pc}\left(\left(\int_0^r \frac{1}{w(\sigma)}d\sigma + 1\right)^{1-pc} d\rho - 1\right) \\
&\le \frac{1}{pc-1}.
\end{aligned}$$

Since $a + b \ge 0$, M_1' is estimated as follows:

$$M_1' \le \frac{1}{pc-1} 2^{(n-1)/p} C^{1/p} (1 + C_0)^{(-a)_+}.$$

M_2' is estimated similarly. Then, the estimate (A.2.1.13) follows from Corollary A.2.1.1. Q.E.D.

In a way similar to the proof of Theorem A.2.1.2, we have the following theorem for $p = \infty$.

Theorem A.2.1.3. *Let $w \in \mathscr{F}$. Let w_0, w_1, w_2 be defined by*

$$\begin{aligned}
w_0(r) &= \Big(\int_0^r \frac{1}{w(\rho)}d\rho + 1\Big)^{-a}, \\
w_1(r) &= \Big(\int_0^r \frac{1}{w(\rho)}d\rho + 1\Big)^{-b}, \\
w_2(r) &= \Big(\int_0^r \frac{1}{w(\rho)}d\rho + 1\Big)^{-c}
\end{aligned}$$

with $a, b, c \in \mathbb{R}$ satisfying

$$a + b + c_- \ge 0 \qquad \text{and} \qquad a + b_- + c \ge 0,$$

where $b_- = -\max(0, -b) = \min(0, b)$, $c_- = -\max(0, -c) = \min(0, c)$.
Then, there exists $C > 0$ such that the trilinear estimate

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} w_0(|x+y|) w_1(|x|) w_2(|y|) |f(x+y) g(x) h(y)| \; dx \; dy \le C \|f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} \|h\|_{L^1(\mathbb{R}^n)}$$

holds for all $f \in L^\infty(\mathbb{R}^n)$, $g, h \in L^1(\mathbb{R}^n)$.

Theorem A.2.1.2 shows the importance of the class $\mathscr{F}$ to the trilinear estimate such as (A.2.1.8). Accordingly, below we study the class $\mathscr{F}$ in details. In Section A.2.2, we study a basic property of $\mathscr{F}$. In Section A.2.3, we introduce arbitrarily and infinitely iterates of logarithm in connection with $\mathscr{F}$. A part of the arguments in Sections 2 and 3 are essentially given by Ando, Horiuchi, and Nakai [1]. We revisit them in the present framework for definiteness. In Section A.2.4, we study optimality of Theorem A.2.1.2.

A.2.2 A Basic Property of $\mathscr{F}$

In this section we prove:

Proposition A.2.2.1. *For $w \in \mathscr{F}$ and $a \in \mathbb{R}$, we define W_a by*

$$W_a(r) = w(r)\left(\int_0^r \frac{1}{w(\rho)}d\rho + 1\right)^a, \qquad r \geq 0.$$

Then, $W_a \in \mathscr{F}$.

proof. By definition, we see that W_a is a positive, continuous function on $[0,\infty)$ satisfying $1/W_a \in L^1_{\mathrm{loc}}(0,\infty)$. By (A2) and Remark A.2.1.1,

$$W_a(2r) \leq Cw(r)\left(\int_0^{2r} \frac{1}{w(\rho)}d\rho + 1\right)^a \leq C(C_0+1)^{a_+}W_a(r),$$

where $a_+ = \max(a,0)$. It remains to prove that W_a satisfies (A1); For any $a, b \in \mathbb{R}$, there exists $C_{a,b}$ such that for any r and R with $0 \leq r \leq R$,

$$W_a(r)\left(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\right)^b \leq C_{a,b}W_a(R)\left(\int_0^R \frac{1}{W_a(\rho)}d\rho + 1\right)^b$$

holds. Let $0 \leq r \leq R$. We note that (A1) property of w is equivalent to $W_a(r) \leq C_a W_a(R)$. We distinguish three cases:

(i) $b \geq 0$. (ii) $b < 0$, $a \geq 0$. (iii) $b < 0$, $a < 0$.

(i) In the case where $b \geq 0$, we estimate

$$\begin{aligned} W_a(r)\left(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\right)^b &\leq C_a W_a(R)\left(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\right)^b \\ &\leq C_a W_a(R)\left(\int_0^R \frac{1}{W_a(\rho)}d\rho + 1\right)^b, \end{aligned}$$

as required.

(ii) In the case where $b < 0$, $a \geq 0$, we first notice that

$$\begin{aligned} &\frac{1}{W_a(R)}\left(\int_0^R \frac{1}{W_a(\rho)}d\rho + 1\right)^{|b|} \\ &= \frac{1}{W_a(R)}\left(\int_0^r \frac{1}{W_a(\rho)}d\rho + \int_r^R \frac{1}{W_a(\rho)}d\rho + 1\right)^{|b|} \\ &\leq \frac{2^{(|b|-1)_+}}{W_a(R)}\left(\left(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\right)^{|b|} + \left(\int_r^R \frac{1}{W_a(\rho)}d\rho\right)^{|b|}\right) \\ &\leq \frac{C_a 2^{(|b|-1)_+}}{W_a(r)}\left(\int_0^r \frac{1}{W_a(\rho)}d\rho + 1\right)^{|b|} + \frac{2^{(|b|-1)_+}}{W_a(R)}\left(\int_r^R \frac{1}{W_a(\rho)}d\rho\right)^{|b|}. \end{aligned} \tag{A.2.2.1}$$

To estimate the second term on the right hand side of the last inequality of (A.2.2.1), we remark that

$$\begin{aligned}\int_r^R \frac{1}{W_a(\rho)} d\rho &= \int_r^R \frac{1}{w(\rho)} \left(\int_0^\rho \frac{1}{w(\sigma)} d\sigma + 1 \right)^{-a} d\rho \\ &\le \int_r^R \frac{1}{w(\rho)} \left(\int_0^r \frac{1}{w(\sigma)} d\sigma + 1 \right)^{-a} d\rho \\ &\le \left(\int_0^r \frac{1}{w(\sigma)} d\sigma + 1 \right)^{-a} \int_0^R \frac{1}{w(\rho)} d\rho\end{aligned}$$

and

$$\frac{1}{W_a(R)} = \frac{1}{w(R)} \left(\int_0^R \frac{1}{w(\rho)} d\rho + 1 \right)^{-a} \le \frac{1}{w(R)} \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^{-a}.$$

Therefore,

$$\begin{aligned}&\frac{1}{W_a(R)} \left(\int_r^R \frac{1}{W_a(\rho)} d\rho \right)^{|b|} \\ &\le \frac{1}{w(R)} \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^{-a-a|b|} \left(\int_0^R \frac{1}{w(\rho)} d\rho \right)^{|b|} \\ &\le \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^{-a-a|b|} \frac{1}{w(R)} \left(\int_0^R \frac{1}{w(\rho)} d\rho + 1 \right)^{|b|} \\ &\le \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^{-a-a|b|} \cdot C_b \frac{1}{w(r)} \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^{|b|} \\ &\le C_b \frac{1}{w(r) \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^a} \cdot \left(\int_0^r \frac{1}{w(\rho) \left(\int_0^r \frac{1}{w(\sigma)} d\sigma + 1 \right)^a} d\rho + 1 \right)^{|b|} \\ &\le C_b \frac{1}{W_a(r)} \left(\int_0^r \frac{1}{W_a(\rho)} d\rho + 1 \right)^{|b|}. \qquad \text{(A.2.2.2)}\end{aligned}$$

Combining (A.2.2.1) and (A.2.2.2) and taking the inverse of the resulting inequality, we find that W_a satisfies (A1).

(iii) In the case where $b < 0$, $a < 0$, we use the equality

$$\int_0^r \frac{1}{W_a(\rho)} d\rho + 1 = \frac{1}{|a|+1} \left(\int_0^r \frac{1}{w(\rho)} d\rho + 1 \right)^{|a|+1} + \frac{|a|}{|a|+1}$$

to estimate

$$\begin{aligned}
&W_a(r)\left(\int_0^r \frac{1}{W_a(\rho)}d\rho+1\right)^b \\
&\le \frac{1}{(|a|+1)^b}w(r)\left(\int_0^r \frac{1}{w(\rho)}+1\right)^{a+(|a|+1)b} \\
&\le (|a|+1)^{|b|}C_{a+b-ab}w(R)\left(\int_0^R \frac{1}{w(\rho)}d\rho+1\right)^{a+b-ab} \\
&= (|a|+1)^{|b|}C_{a+b-ab}W_a(R)\left(\int_0^R \frac{1}{w(\rho)}d\rho+1\right)^{(|a|+1)b} \\
&\le (|a|+1)^{|b|}C_{a+b-ab}W_a(R)\left(\int_0^R \frac{1}{W_a(\rho)}d\rho+1\right)^b,
\end{aligned}$$

as required. Q.E.D.

A.2.3 Infinitely Iterated Logarithm

In this section, we introduce arbitrarily and infinitely iterated logarithm functions in connection with class $\mathscr{F}$. The definition is different from that of [1] in the sense that convergence factors are introduced in terms of the parameter $\theta \in (0, 1]$.

Definition A.2.3.1. *Let $0 < \theta \le 1$. For non-negative integers n, the following functions $l_{\theta,n} : [0, \infty) \to \mathbb{R}$ are defined successively by:*

$$\begin{aligned}
&l_{\theta,0}(r) = 1 + r, \\
&l_{\theta,k}(r) = 1 + \theta \log l_{\theta,k-1}(r), \quad k \ge 1.
\end{aligned}$$

Moreover, we define $L_{\theta,k} : [0, \infty) \to \mathbb{R}$ by

$$L_{\theta,k}(r) = \prod_{j=0}^{k} l_{\theta,j}(r).$$

Remark A.2.3.2. *For any $k \ge 0$, $l_{\theta,k}(0) = L_{\theta,k}(0) = 1$. Moreover, $l_{\theta,k}(r) \ge 1$ and $L_{\theta,k}(r) \ge 1$ for all $r \ge 0$ since $l_{\theta,k}$ and $L_{\theta,k}$ are increasing functions. Explicitly, the derivative $l'_{\theta,k}$ is given by*

$$l'_{\theta,k}(r) = \theta^k \cdot \frac{1}{L_{\theta,k-1}(r)}, \qquad r \ge 0.$$

By a successive use of the elementary inequality $\log(1 + r) \le r$ *for* $r \ge -1$,

$$0 \le \log l_{\theta,k}(r) \le \theta \log l_{\theta,k-1}(r) \le \cdots \le \theta^k \log l_{\theta,0}(r), \quad r \ge 0.$$

This implies that for any θ with $0 < \theta < 1$, the series $\sum_{k=0}^{\infty} \log l_{\theta,k}(r)$ converges with estimates

$$0 \le \sum_{k=0}^{\infty} \log l_{\theta,k}(r) \le \frac{1}{1-\theta} \log l_{\theta,0}(r), \quad r \ge 0.$$

Definition A.2.3.3. *For any θ with $0 < \theta < 1$, L_θ is defined by*

$$L_\theta(r) = \prod_{k=0}^{\infty} l_{\theta,k}(r), \qquad r \geq 0.$$

Remark A.2.3.4. *By Remark A.2.3.2, if $0 < \theta < 1$, L_θ converges with estimates*

$$1 \leq L_\theta(r) \leq (1+r)^{1/(1-\theta)}, \quad r \geq 0.$$

If $\theta = 1$ and $r > 0$, we prove that $L_1(r) = \infty$ by contradiction. Assume that $L_1(r) < \infty$. Then, for any k we have

$$\begin{aligned} \log L_1(r) &\geq \log L_{1,k}(r) \\ &= \int_0^r \frac{d}{d\rho}\left(\sum_{j=0}^{k} \log l_{1,j}(\rho)\right) d\rho \\ &= \int_0^r \sum_{j=0}^{k} \frac{1}{L_{1,j}(\rho)} d\rho \\ &\geq \int_0^r \sum_{j=0}^{k} \frac{1}{L_{1,k}(r)} d\rho = r \sum_{j=0}^{k} \frac{1}{L_{1,k}(r)} \geq \frac{(k+1)r}{L_1(r)}, \end{aligned}$$

which yields a contradiction for k sufficiently large.

The main theorem in this section now reads:

Theorem A.2.3.1. *For any θ with $0 < \theta < 1$, $L_\theta \in \mathscr{F}$. Moreover,*

$$\int_0^\infty \frac{1}{L_\theta(r)} dr = \infty. \tag{A.2.3.1}$$

To prove Theorem A.2.3.1, we introduce some preliminary propositions. From now on, θ denotes a real number with $0 < \theta < 1$ without particular comments.

Lemma A.2.3.1. *For any $a \in \mathbb{R}$, there exists $C_{\theta,a} \geq 1$ such that for any r and R with $0 \leq r \leq R$*

$$(1+r)\left(\int_0^r \frac{1}{L_\theta(\rho)} d\rho + 1\right)^a \leq C_{\theta,a}(1+R)\left(\int_0^R \frac{1}{L_\theta(\rho)} d\rho + 1\right)^a \tag{A.2.3.2}$$

holds.

proof. For $a \geq 0$, (A.2.3.2) holds with $C_a = 1$ by monotonicity. Let $a < 0$ and let m_θ be defined by

$$m_\theta(r) = \int_0^r \frac{1}{L_\theta(\rho)} d\rho + 1.$$

Then,

$$m_\theta'(R) = \frac{1}{L_\theta(R)} \leq \frac{m_\theta(r)}{l_{\theta,1}(R) l_{\theta,0}(R)} \leq \frac{m_\theta(r)}{\theta l_{\theta,1}(r)} l_{\theta,1}'(R). \tag{A.2.3.3}$$

By (A.2.3.3), we have

$$\begin{aligned} m_\theta(R) &= m_\theta(r) + \int_r^R m'_\theta(\rho)d\rho \\ &\le m_\theta(r) + \frac{m_\theta(r)}{\theta l_{\theta,1}(r)} \int_r^R l'_{\theta,1}(\rho)d\rho \\ &= m_\theta(r) + \frac{m_\theta(r)}{\theta l_{\theta,1}(r)}\Big(l_{\theta,1}(R) - l_{\theta,1}(r)\Big) \\ &\le \frac{m_\theta(r)}{\theta l_{\theta,1}(r)} l_{\theta,1}(R). \end{aligned} \tag{A.2.3.4}$$

By Remark A.2.3.2 and (A.2.3.4), we obtain

$$\begin{aligned} (1+r)m_\theta(r)^a &= \left(\frac{m_\theta(r)}{l_{\theta,1}(r)}\right)^a (1+r)\Big(l_{\theta,1}(r)\Big)^a \\ &\le C\left(\frac{m_\theta(r)}{l_{\theta,1}(r)}\right)^a (1+R)\Big(l_{\theta,1}(R)\Big)^a \\ &\le C\theta^a(1+R)m_\theta(R)^a \end{aligned}$$

with some constant C, as required. Q.E.D.

Lemma A.2.3.2. *For any* $r, s \ge 0$,

$$L_\theta(l_{\theta,0}(s)r) \le L_\theta(s)L_\theta(r). \tag{A.2.3.5}$$

proof. It is sufficient to prove that

$$l_{\theta,k}(l_{\theta,0}(s)r) \le l_{\theta,k}(s)l_{\theta,k}(r) \tag*{$(A.2.3.6)_k$}$$

by induction on $k \ge 0$. For $k = 0$,

$$l_{\theta,0}(l_{\theta,0}(s)r) = 1 + l_{\theta,0}(s)r = 1 + (1+s)r \le (1+s)(1+r) = l_{\theta,0}(s)l_{\theta,0}(r)$$

Let $k \ge 1$ and assume $(\text{A.2.3.6})_{k-1}$. Then,

$$\begin{aligned} l_{\theta,k}(l_{\theta,0}(s)r) &= 1 + \theta\log\Big(l_{\theta,k-1}(l_{\theta,0}(s)r)\Big) \\ &\le 1 + \theta\log\Big(l_{\theta,k-1}(s)l_{\theta,k-1}(r)\Big) \\ &\le \Big(1+\theta\log l_{\theta,k-1}(s)\Big)\Big(1+\theta\log l_{\theta,k-1}(r)\Big) \\ &\le l_{\theta,k}(s)l_{\theta,k}(r), \end{aligned}$$

which completes the induction argument. Q.E.D.

Lemma A.2.3.3. *For any non-negative integers* k *and* j, $l_{\theta,k+j}$ *is represented by* $l_{\theta,k}$ *and* $l_{\theta,j}$ *as*

$$l_{\theta,k+j}(r) = l_{\theta,j}\Big(l_{\theta,k}(r) - 1\Big) \tag{A.2.3.7}$$

for all $r \ge 0$.

proof. We prove (A.2.3.7) by induction on j. For $j = 0$, we have

$$l_{\theta,k}(r) = l_{\theta,0}\Big(l_{\theta,k}(r) - 1\Big)$$

for all $k \geq 0$ by definition. Let $j \geq 1$ and assume that

$$l_{\theta,k+j-1}(r) = l_{\theta,j-1}\Big(l_{\theta,k}(r) - 1\Big)$$

holds for all $k \geq 0$ and $r \geq 0$. Then,

$$\begin{aligned} l_{\theta,k+j}(r) &= 1 + \theta \log\Big(l_{\theta,k+j-1}(r)\Big) \\ &= 1 + \theta \log\Big(l_{\theta,j-1}(l_{\theta,k}(r) - 1)\Big) \\ &= l_{\theta,j}\Big(l_{\theta,k}(r) - 1\Big) \end{aligned}$$

for all $k \geq 0$ and $r \geq 0$. This completes the induction argument. Q.E.D.

Proof of Theorem A.2.3.1. Let r, R satisfy $0 \leq r \leq R$. Then, by Lemma A.2.3.1,

$$\begin{aligned} L_\theta(r)\left(\int_0^r \frac{1}{L_\theta(\rho)} d\rho + 1\right)^a &\leq (1+r)\left(\prod_{k=1}^{\infty} l_{\theta,k}(R)\right)\left(\int_0^r \frac{1}{L_\theta(\rho)} d\rho + 1\right)^a \\ &\leq C_{\theta,a} L_\theta(R)\left(\int_0^R \frac{1}{L_\theta(\rho)} d\rho + 1\right)^a. \end{aligned}$$

Moreover, since $l_{\theta,0}(1) = 2$, we apply (A.2.3.5) with $s = 1$ to obtain

$$L_\theta(2r) \leq L_\theta(1) L_\theta(r).$$

Therefore, $L_\theta \in \mathscr{F}$. We prove (A.2.3.1). It suffices to prove that there exists a sequence $\{r_k; k \geq 0\}$ of positive numbers such that

$$\int_0^{r_k} \frac{1}{L_\theta(\rho)} d\rho \to \infty$$

as $k \to \infty$. Let $r_0 = 1$. Then, for any $k \geq 1$ there exists a unique $r_k > 0$ such that $l_{\theta,k}(r_k) = l_{\theta,0}(r_0) = 2$ since $l_{\theta,k}$ is an increasing function with $l_{\theta,k}(0) = 1$ and $\lim_{r\to\infty} l_{\theta,k}(r) = \infty$. Let $0 \leq \rho \leq r_k$. By Lemma A.2.3.3,

$$\begin{aligned} L_\theta(\rho) &= L_{\theta,k-1}(\rho) \prod_{j=0}^{\infty} l_{\theta,k+j}(\rho) \\ &\leq L_{\theta,k-1}(\rho) \prod_{j=0}^{\infty} l_{\theta,k+j}(r_k) \\ &= L_{\theta,k-1}(\rho) \prod_{j=0}^{\infty} l_{\theta,j}\Big(l_{\theta,k}(r_k) - 1\Big) \\ &= L_{\theta,k-1}(\rho)\, L_\theta\Big(l_{\theta,k}(r_k) - 1\Big) \\ &= L_{\theta,k-1}(\rho)\, L_\theta\Big(l_{\theta,0}(r_0) - 1\Big) \\ &= L_{\theta,k-1}(\rho)\, L_\theta(1). \end{aligned} \tag{A.2.3.8}$$

By (A.2.3.7) and (A.2.3.8),

$$\begin{aligned}\int_0^{r_k} \frac{1}{L_\theta(\rho)} d\rho &\geq \frac{1}{L_\theta(1)} \int_0^{r_k} \frac{1}{L_{\theta,k-1}(\rho)} d\rho \\ &= \frac{1}{L_\theta(1)} \frac{1}{\theta^k} \Big(l_{\theta,k}(r_k) - 1 \Big) \\ &= \frac{1}{L_\theta(1)} \frac{1}{\theta^k} \to \infty \end{aligned}$$

as $k \to \infty$, as required. Q.E.D.

A.2.4 Optimality of Theorems A.2.1.2 and A.2.1.3.

In this section, we consider optimality of Theorems A.2.1.2 and A.2.1.3. To this end, we divide weight functions $w \in \mathscr{F}$ into two cases:

$$\mathrm{I} : \int_0^\infty \frac{1}{w(r)} dr < \infty. \qquad \mathrm{II} : \int_0^\infty \frac{1}{w(r)} dr = \infty.$$

Theorem A.2.4.1. *Let $2 \leq p < \infty$ and let $w \in \mathscr{F}$. Let w_0, w_1, w_2 be as in Theorem A.2.1.2 with a, b, $c \in \mathbb{R}$.*

(1) In the case I, the trilinear estimate in Theorem A.2.1.2 holds for any a, b, $c \in \mathbb{R}$.

(2) In the case II, let a, b, c satisfy one of the conditions (iii), (iv), (v), (vi):

(iii) $a + b + c < 1/p$. (iv) $a + b < 0$. (v) $a + c < 0$.

(vi) $a + b + c = 1/p$ and "$a + b = 0$ or $a + c = 0$".

Then, the trilinear estimate in Theorem A.2.1.2 fails for some $f \in L^p(\mathbb{R}^n)$, g, $h \in L^{p'}(\mathbb{R}^n)$.

Remark A.2.4.1. *The conditions (iii), (iv), (v), and (vi) in Theorem A.2.4.1 consist of the negation of the condition "(i) or (ii)" in Theorem A.2.1.2.*

proof. In the case I, we easily see the trilinear estimate holds with any a, b, and c. To give a counter example for the trilinear estimate in the case II, we divide the proof into three cases:

(i) $a + b + c < 1/p$. (ii) $a + b < 0$ or $a + c < 0$.

(iii) $a + b + c = 1/p$ and "$a + b = 0$ or $a + c = 0$".

(i) In the case where $a + b + c < 1/p$, let $\delta > 0$ satisfy $\delta \neq 1/p - c$ and let

$$\begin{aligned} f(x) &= (1 + |x|)^{-(n-1)/p} w(|x|)^{-1/p} \left(\int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-\delta}, \\ g(x) = h(x) &= (1 + |x|)^{-(n-1)/p'} w(|x|)^{-1/p'} \left(\int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p'-\delta}. \end{aligned}$$

Then, $f \in L^p(\mathbb{R}^n)$ and $g, h \in L^{p'}(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$ with $|x| \geq 2$,

$$\begin{aligned}
&\int_{1\leq|y|\leq|x|/2} w_0(|x+y|)f(x+y)w_2(|y|)h(y)dy \\
&= \int_{1\leq|y|\leq|x|/2} \left(\int_0^{|x+y|} \frac{1}{w(\rho)}d\rho + 1\right)^{-1/p-a-\delta} \\
&\quad \cdot (1+|y|)^{-(n-1)}\frac{1}{w(|y|)}\left(\int_0^{|y|} \frac{1}{w(\rho)}d\rho + 1\right)^{-1/p'-c-\delta} dy. \qquad (A.2.4.1)
\end{aligned}$$

By (A1), if $1/p + a + \delta \geq 0$, then for any $y \in \mathbb{R}^n$ with $0 \leq |y| \leq |x|/2$,

$$\begin{aligned}
&\left(\int_0^{|x+y|} \frac{1}{w(\rho)}d\rho + 1\right)^{-1/p-a-\delta} \\
&\geq \left(\int_0^{3|x|/2} \frac{1}{w(\rho)}d\rho + 1\right)^{-1/p-a-\delta} \\
&= \left(\frac{3}{2}\int_0^{|x|} \frac{1}{w(3\rho/2)}d\rho + 1\right)^{-1/p-a-\delta} \\
&\geq \left(\frac{3C_0}{2}\int_0^{|x|} \frac{1}{w(\rho)}d\rho + 1\right)^{-1/p-a-\delta} \\
&\geq \left(\frac{3C_0}{2} + 1\right)^{-1/p-a-\delta}\left(\int_0^{|x|} \frac{1}{w(\rho)}d\rho + 1\right)^{-1/p-a-\delta}. \qquad (A.2.4.2)
\end{aligned}$$

Similarly, if $1/p + a + \delta < 0$, then for any $y \in \mathbb{R}^n$ with $0 \leq |y| \leq |x|/2$,

$$\begin{aligned}
&\left(\int_0^{|x+y|} \frac{1}{w(\rho)}d\rho + 1\right)^{-1/p-a-\delta} \\
&\geq \left(\int_0^{|x|/2} \frac{1}{w(\rho)}d\rho + 1\right)^{-1/p-a-\delta} \\
&= \left(\frac{1}{2}\int_0^{|x|} \frac{1}{w(\rho/2)}d\rho + 1\right)^{-1/p-a-\delta} \\
&\geq \left(\frac{1}{2C_0}\int_0^{|x|} \frac{1}{w(\rho)}d\rho + 1\right)^{-1/p-a-\delta} \\
&\geq (2C_0)^{1/p+a+\delta}\left(\int_0^{|x|} \frac{1}{w(\rho)}d\rho + 1\right)^{-1/p-a-\delta}. \qquad (A.2.4.3)
\end{aligned}$$

In addition, if $1/p - c - \delta \geq 0$, then for any $x \in \mathbb{R}^n$ with $|x| \geq 4$,

$$\begin{aligned}
&\int_{1\leq|y|\leq|x|/2}(1+|y|)^{-(n-1)}\frac{1}{w(|y|)}\left(\int_0^{|y|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'-c-\delta}dy\\
&=\omega_{n-1}\int_1^{|x|/2}\left(\frac{r}{1+r}\right)^{n-1}\frac{1}{w(r)}\left(\int_0^{r}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'-c-\delta}dr\\
&\geq\frac{2^{1-n}\omega_{n-1}}{1/p-c-\delta}\left(\left(\int_0^{|x|/2}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}-\left(\int_0^{1}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}\right)\\
&\geq\frac{2^{1-n}\omega_{n-1}}{1/p-c-\delta}\left(1-\left(\int_0^{2}\frac{1}{w(r)}dr+1\right)^{-1/p+c+\delta}\right)\\
&\quad\cdot\left(\int_0^{|x|/2}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}\\
&=\frac{2^{1-n}\omega_{n-1}}{1/p-c-\delta}\left(1-\left(\int_0^{2}\frac{1}{w(r)}dr+1\right)^{-1/p+c+\delta}\right)\\
&\quad\cdot\left(\frac{1}{2}\int_0^{|x|}\frac{1}{w(r/2)}dr+1\right)^{1/p-c-\delta}\\
&\geq\frac{2^{1/p'-c-\delta-n}\omega_{n-1}}{(1/p-c-\delta)C_0^{1/p-c-\delta}}\left(1-\left(\int_0^{2}\frac{1}{w(r)}dr+1\right)^{-1/p+c+\delta}\right)\\
&\quad\cdot\left(\int_0^{|x|}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}. \qquad \text{(A.2.4.4)}
\end{aligned}$$

If $1/p - c - \delta < 0$, then

$$\begin{aligned}
&\int_{1\leq|y|\leq|x|/2}(1+|y|)^{-(n-1)}\frac{1}{w(|y|)}\left(\int_0^{|y|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'-c-\delta}dy\\
&=\frac{2^{1-n}\omega_{n-1}}{1/p-c-\delta}\left(\left(\int_0^{1}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}-\left(\int_0^{|x|/2}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}\right)\\
&\geq\frac{2^{1-n}\omega_{n}}{1/p-c-\delta}\left(\left(\int_0^{1}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}-\left(\int_0^{2}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}\right)\\
&\quad\cdot\left(\int_0^{|x|}\frac{1}{w(r)}dr+1\right)^{1/p-c-\delta}. \qquad \text{(A.2.4.5)}
\end{aligned}$$

By (A.2.4.1), (A.2.4.2), (A.2.4.3), (A.2.4.4), and (A.2.4.5), there exists a positive constant C such that for any $x \in \mathbb{R}^n$ with $|x| \geq 4$

$$\begin{aligned}
&\int_{1\leq|y|\leq|x|/2}w_0(|x+y|)f(x+y)w_2(|y|)h(y)dy\\
&\geq C\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-a-c-2\delta}. \qquad \text{(A.2.4.6)}
\end{aligned}$$

Finally by (A.2.4.6), we have

$$\begin{aligned}
&\iint w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\,dx\,dy \\
&\geq C\int_{|x|\geq 4}(|x|+1)^{-(n-1)}\frac{1}{w(|x|)}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'-a-b-c-3\delta}dx \\
&\geq C\omega_n\left(\frac{4}{5}\right)^{n-1}\int_4^{\infty}\frac{1}{w(r)}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-1/p'-a-b-c-3\delta}dr \\
&\geq C\omega_n\left(\frac{4}{5}\right)^{n-1}\left(\log\left(\int_0^{\infty}\frac{1}{w(\rho)}d\rho+1\right)-\log\left(\int_0^{4}\frac{1}{w(\rho)}d\rho+1\right)\right) \\
&=\infty
\end{aligned}$$

with $\delta \leq (1/p-a-b-c)/3$.

(ii) In the case where $a+b<0$ or $a+c<0$, by symmetry, it is sufficient to give a counter example only in the case where $a+b<0$. Let f and g be as in the case (iii) with $\delta \leq -(a+b)/2$ and $a+1/p+\delta \neq 1$. Let

$$h(x) = \chi_{B(1)}(x)\frac{1}{w_2(|x|)}.$$

Then, by (A.2.1.15), (A.2.4.4), and (A.2.4.5),

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} w_0(|x+y|)w_1(|x|)w_2(|y|)f(x+y)g(x)h(y)\;dy\,dx \\
&\geq \int_{|x|\geq 2}\int_{|y|\leq 1}\left(\int_0^{|x+y|}\frac{1}{w(\rho)}d\rho+1\right)^{-a-1/p-\delta}dy \\
&\qquad\cdot(1+|x|)^{n-1}\frac{1}{w(|x|)}\left(\int_0^{|x|}\frac{1}{w(\rho)}d\rho+1\right)^{-b-1/p'-\delta}dx \\
&\geq C\int_2^{\infty}\frac{1}{w(r)}\left(\int_0^{r}\frac{1}{w(\rho)}d\rho+1\right)^{-a-b-1-2\delta}dr \\
&\geq C\int_2^{\infty}\frac{1}{w(r)}\left(\int_0^{r}\frac{1}{w(\rho)}d\rho+1\right)^{-1}dr \\
&\geq C\left(\log\left(\int_0^{\infty}\frac{1}{w(\rho)}d\rho+1\right)-\log\left(\int_0^{2}\frac{1}{w(\rho)}d\rho+1\right)\right) \\
&=\infty
\end{aligned}$$

with some positive constant C, as required.

(iii) In the case where $a+b+c=1/p$ and $a+b=0$ or $a+b=c$, by symmetry, it is sufficient to give a counter example in the case where $a+b=0$. Let

$$\begin{aligned}
J(r) &= \int_0^r \frac{1}{w(\rho)}\Big(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\Big)^{-1} d\rho + 1,\\
f(x) &= (1+|x|)^{-(n-1)/p} w(|x|)^{-1/p}\Big(\int_0^{|x|} \frac{1}{w(\rho)}d\rho + 1\Big)^{-1/p} J(|x|)^{-1/p-\delta},\\
g(x) &= h(x)\\
&= (1+|x|)^{-(n-1)/p'} w(|x|)^{-1/p'}\Big(\int_0^{|x|} \frac{1}{w(\rho)}d\rho + 1\Big)^{-1/p'} J(|x|)^{-1/p'-\delta}
\end{aligned}$$

for $\delta > 0$. By (A1),

$$\begin{aligned}
J(2r) &= \int_0^{2r} \frac{1}{w(\rho)}\Big(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\Big)^{-1} d\rho + 1\\
&\le \int_0^{r} \frac{1}{w(\rho)}\Big(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\Big)^{-1} d\rho + 1\\
&+ \int_0^{r} \frac{1}{w(r+\rho)}\Big(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\Big)^{-1} d\rho + 1\\
&\le (1+C_0)J(r). \qquad \text{(A.2.4.7)}
\end{aligned}$$

In addition, with any $k \ge 0$ let $r_k > 0$ satisfy

$$\int_0^{r_k} \frac{1}{w(\rho)}d\rho = 2^k - 1,$$

where r_k is determined uniquely, since $\int_0^r 1/w(\rho)d\rho$ is a monotone increasing function of r. Then, we estimate

$$\begin{aligned}
J(r_k) &= \sum_{j=1}^k \int_{r_{j-1}}^{r_j} \frac{1}{w(\rho)}\Big(\int_0^\rho \frac{1}{w(\sigma)}d\sigma + 1\Big)^{-1} d\rho\\
&\ge \sum_{j=1}^k \Big(\int_0^{r_j} \frac{1}{w(\sigma)}d\sigma + 1\Big)^{-1}\Big(\int_0^{r_j} \frac{1}{w(\rho)}d\rho - \int_0^{r_{j-1}} \frac{1}{w(\rho)}d\rho\Big)\\
&= \sum_{j=1}^k 2^{-j}(2^j - 2^{j-1}) = \frac{k}{2}. \qquad \text{(A.2.4.8)}
\end{aligned}$$

This shows $\lim_{r\to\infty} J(r) = \infty$. By (A.2.4.4), (A.2.4.5), and (A.2.4.7), for any $x \in \mathbb{R}^n$ with $|x| \ge 4$ and

$0 < \delta < 1/p$

$$
\begin{aligned}
&\int_{1\le|y|\le|x|/2} w_0(|x+y|)f(x+y)w_2(|y|)h(y)dy \\
&= \int_{1\le|y|\le|x|/2} \left(\int_0^{|x+y|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a} J(|x+y|)^{-1/p-\delta} \\
&\qquad \cdot (1+|y|)^{-(n-1)} \frac{1}{w(|y|)} \left(\int_0^{|y|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1} J(|y|)^{-1/p'-\delta} dy \\
&\ge C \left(\int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a} J(2|x|)^{-1/p-\delta} \\
&\qquad \cdot \int_1^{|x|/2} \frac{1}{w(r)} \left(\int_0^{r} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1} J(r)^{-1/p'-\delta} dr \\
&\ge C(1+C_0)^{-1/p-\delta} \frac{1}{1/p-\delta} \left(\int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a} J(|x|)^{-1/p-\delta} \\
&\quad \cdot \left(J(|x|)^{1/p-\delta} - J(1)^{1/p-\delta} \right) \\
&\ge C(1+C_0)^{-1/p-\delta} \frac{1}{1/p-\delta} \left(1 - J(2)^{\delta-1/p} \right) \left(\int_0^{|x|} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1/p-a} \\
&\quad \cdot J(|x|)^{-2\delta}
\end{aligned} \tag{A.2.4.9}
$$

with some positive constant C. Then, by (A.2.4.9) and (A.2.4.8), for $0 < \delta \le 1/(3p)$, we estimate

$$
\begin{aligned}
&\iint_{\mathbb{R}^n \times \mathbb{R}^n} w_0(|x+y|)w_1(|x|)w_2(|y|)f(x+y)g(x)h(y)\ dy\, dx \\
&\ge \int_{|x|\ge 4} \int_{1\le|y|\le|x|/2} w_0(|x+y|)w_1(|x|)w_2(|y|)f(x+y)g(x)h(y)\ dy\, dx \\
&\ge C(1+C_0)^{-1/p-\delta} \frac{1}{1/p-\delta} \left(1 - J(2)^{\delta-1/p} \right) \\
&\quad \cdot \int_4^{\infty} \frac{1}{w(r)} \left(\int_0^{r} \frac{1}{w(\rho)} d\rho + 1 \right)^{-1} J(|x|)^{-1/p'-3\delta} dr \\
&\ge C(1+C_0)^{-1/p-\delta} \frac{1}{1/p-\delta} \left(1 - J(2)^{\delta-1/p} \right) \left(\lim_{r\to\infty} \log J(r) - \log J(4) \right) \\
&= \infty,
\end{aligned}
$$

as required. Q.E.D.

Theorem A.2.4.2. *Let $w \in \mathscr{F}$ and let w_0, w_1, w_2 be as in Theorem A.2.1.2 with a, b, $c \in \mathbb{R}$.*

(1) In the case I, the trilinear estimate in Theorem A.2.1.2 holds for any a, b, $c \in \mathbb{R}$

(2) In the case II, let a, b, c satisfy either (iii) or (iv) or (v) in Theorem A.2.4.1, then the trilinear estimate in Theorem A.2.1.2 fails for some $f \in L^\infty(\mathbb{R}^n)$, g, $h \in L^1(\mathbb{R}^n)$.

(3) In the case II, let $a = b = c = 0$. Then, the trilinear estimates holds.

proof. The proofs of (1) and (2) are the same as in the proof of Theorem A.2.4.1, while (3) follows from the Hölder and Young inequalities as below:

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\ dx\,dy \\
&\le \iint_{\mathbb{R}^n\times\mathbb{R}^n} |f(x+y)g(x)h(y)|\ dx\,dy \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} |f(x)g(x-y)h(y)|\ dy\,dx \\
&\le \|f\|_{L^\infty(\mathbb{R}^n)}\|g*h\|_{L^1(\mathbb{R}^n)} \\
&\le \|f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}\|h\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Q.E.D.

Chapter A.3

Study of Fractional Leibniz Rule

A.3.1 Introduction

Here, we revisit Lemma 1.3.11 with $-a = b = c = s$ from the view point of the remainder of main contribution.

One of the most important tools to obtain local well-posedness of nonlinear equations of mathematical physics is based on the bilinear estimate of the form

$$\|D^s(fg)\|_{L^p(\mathbb{R}^n)} \leq C\|D^s f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)} + C\|f\|_{L^{p_3}}\|D^s g\|_{L^{p_4}(\mathbb{R}^n)}, \tag{A.3.1.1}$$

where $D^s = (-\Delta)^{s/2}$ is the standard Riesz potential of order $s \in \mathbb{R}$ and $f, g \in \mathscr{S}(\mathbb{R}^n)$. A typical domain for parameters $s, p, p_j, j = 1, \cdots, 4$, where (A.3.1.1) is valid is

$$s > 0, \quad 1 < p, p_1, p_2, p_3, p_4 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Classical proof can be found in [44]. The estimate can be considered as natural homogeneous version of the non-homogeneous inequality of type (A.3.1.1) involving Bessel potentials $(1-\Delta)^{s/2}$ in the place of D^s, obtained by Kato and Ponce in [57] (for this the estimates of type (A.3.1.1) are called Kato-Ponce estimates, too). More general domain for parameters can be found in [41].

Another estimate showing the flexibility in the redistribution of fractional derivatives can be deduced when $0 < s < 1$. More precisely, Kenig, Ponce, and Vega [58] obtained the estimate

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^p(\mathbb{R}^n)} \leq C\|D^{s_1} f\|_{L^{p_1}(\mathbb{R}^n)}\|D^{s_2} g\|_{L^{p_2}(\mathbb{R}^n)}, \tag{A.3.1.2}$$

provided

$$0 < s = s_1 + s_2 < 1, \quad s_1, s_2 \geq 0,$$

and

$$1 < p, p_1, p_2 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \tag{A.3.1.3}$$

One can interpret the bilinear form

$$\mathrm{Cor}_s(f, g) = fD^s g + gD^s f$$

as a correction term such that for any redistribution of the order s of the derivatives, i.e. for any $s_1, s_2 \geq 0$, such that $s_1 + s_2 = s$, we have

$$\|D^s(fg) - \mathrm{Cor}_s(f, g)\|_{L^p(\mathbb{R}^n)} \leq C\|D^{s_1} f\|_{L^{p_1}(\mathbb{R}^n)}\|D^{s_2} g\|_{L^{p_2}(\mathbb{R}^n)}, \tag{A.3.1.4}$$

i.e., we have flexible redistribution of the derivatives of the remainder $D^s(fg) - \mathrm{Cor}_s(f,g)$.

Estimates of the form (A.3.1.2) are of interest on their own in harmonic analysis [4, 5, 7, 12, 19, 25, 40, 42, 43, 44, 45, 57, 75, 91] as well as in applications to nonlinear partial differential equations [8, 23, 27, 47, 58, 61, 68, 80, 82]. Our goal is to generalize (A.3.1.2) in the case where $s \geq 1$. In fact, for $s = 2$, we have $D^2 = -\Delta$ and

$$D^2(fg) - fD^2g - gD^2f + 2\nabla f \cdot \nabla g = 0.$$

This means that we could expect (A.3.1.4) with appropriate correction terms in a general setting.

Typically, one can use paraproduct decomposition and reduce the proof of (A.3.1.4) separating different frequency domains for the supports of $\widehat{f}$ and $\widehat{g}$. In the case, when $\widehat{f}$ is localized in low-frequency domain and $\widehat{g}$ is localized in high-frequency domain, the estimate (A.3.1.4) can be derived from the commutator estimate

$$\|[D^s, f]g\|_{L^p(\mathbb{R}^n)} \leq C\|D^{s_1}f\|_{L^{p_1}(\mathbb{R}^n)}\|D^{s_2}g\|_{L^{p_2}(\mathbb{R}^n)},$$

where the assumption $s \leq 1$ plays a crucial role. More precisely, if we assume

$$\mathrm{supp}\ \widehat{f} \subset \{\xi \in \mathbb{R}^n;\ |\xi| \leq 2^{k-2}\},\quad \mathrm{supp}\ \widehat{g} \subset \{\xi \in \mathbb{R}^n;\ 2^{k-1} \leq |\xi| \leq 2^{k+1}\}, \tag{A.3.1.5}$$

then we can use the relation

$$[D^s, f]g(x) = A_s(Df, D^{s-1}g)(x),$$

where

$$A_s(F,G)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi+\eta)} a_s(\xi,\eta)\widehat{F}(\xi)\widehat{G}(\eta)d\xi d\eta$$

is a Coifman-Meyer type bilinear operator with a symbol $a_s(\xi,\eta)$ of Coifman-Meyer class supported in the cone

$$\Gamma = \{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n; 0 < |\xi| \leq |\eta|/2\}, \tag{A.3.1.6}$$

Recall the definition of Coifman-Meyer class:

Definition A.3.1.1. *We say that a symbol*

$$\sigma \in C^\infty(\mathbb{R}^n \setminus \{0\})$$

belongs to the Hörmander class S^0, *if for all multi-indices* $\alpha \in \mathbb{N}_0^n$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, *we have*

$$|\partial_\xi^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}, \quad \forall \xi \neq 0.$$

We say that a bilinear symbol

$$a \in C^\infty((\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(0,0)\})$$

belongs to the Coifman-Meyer (CM) class, if

$$|\partial_\xi^\alpha \partial_\eta^\beta a(\xi,\eta)| \leq C_{\alpha,\beta}(|\xi| + |\eta|)^{-|\alpha|-|\beta|}.$$

for all multi-indices $\alpha, \beta : |\alpha| + |\beta| < m_n$, *where* m_n *depends on the dimension only.*

It is well-known that operators with symbols in S^0 give rise to bounded operators on $L^p(\mathbb{R}^n) : 1 < p < \infty$ spaces. The result of Coifman and Meyer (see [24, 26, 38, 59]) generalizes this result to bilinear symbols. Namely, it states that bilinear operators

$$A(F,G)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi+\eta)} a(\xi,\eta)\widehat{F}(\xi)\widehat{G}(\eta)d\xi d\eta$$

with symbols in the CM class satisfy

$$\|A(F,G)\|_{L^p(\mathbb{R}^n)} \le C_{p,p_1,p_2}\|F\|_{L^{p_1}(\mathbb{R}^n)}\|G\|_{L^{p_2}(\mathbb{R}^n)} \tag{A.3.1.7}$$

for all $1 < p, p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$.

Applying Coifman-Meyer bilinear estimate for A_s we can deduce the following estimate

Lemma A.3.1.1. *Suppose f, g satisfy the assumptions* (A.3.1.5) *and p, p_1, p_2 satisfy $1 < p, p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. Then for any $s \ge 0$ we have*

$$\|[D^s, f]g\|_{L^p(\mathbb{R}^n)} \le C\|D^1 f\|_{L^{p_1}(\mathbb{R}^n)}\|D^{s-1}g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{A.3.1.8}$$

This estimate and the assumptions (A.3.1.5) explains the possibility to redistribute the fractional derivatives. Namely, if f and g satisfy (A.3.1.5), we have the possibility to replace the right hand side of (A.3.1.8) by $C\|D^{s_1}f\|_{L^{p_1}(\mathbb{R}^n)}\|D^{s_2}g\|_{L^{p_2}(\mathbb{R}^n)}$ for any couple (s_1, s_2) of non-negative real numbers with $0 < s_1 + s_2 = s < 1$.

Our main goal is to study a similar effect of arbitrary redistribution of fractional derivatives for $s \ge 2$ in the scale of Lebesgue and Triebel-Lizorkin spaces in $\mathbb{R}^n$.

First, we shall try to explain the correction term in (A.3.1.4), such that estimate of type

$$\|[D^s, f]g - \mathrm{Cor}_s(f,g)\|_{L^p(\mathbb{R}^n)} \le C\|D^2 f\|_{L^{p_1}(\mathbb{R}^n)}\|D^{s-2}g\|_{L^{p_2}(\mathbb{R}^n)}$$

will be fulfilled.

Let $a_s(\xi,\eta,\theta) = |\eta + \theta\xi|^s$. We also define

$$\begin{aligned} A_s^m(\theta)(f,g) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi+\eta)}\frac{1}{m!}\partial_\theta^m a_s(\xi,\eta,\theta)\hat{f}(\xi)\hat{g}(\eta)d\xi d\eta, \\ \widetilde{A}_s^\alpha(\theta)(f,g) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi+\eta)}\frac{\alpha!}{|\alpha|!}\partial_\eta^\alpha a_s(\xi,\eta,\theta)\hat{f}(\xi)\hat{g}(\eta)d\xi d\eta. \end{aligned} \tag{A.3.1.9}$$

Then $A_s^0(1)(f,g) = D^s(fg)$, $A_s^0(0)(f,g) = fD^s g$, and $A_s^1(0)(f,g) = s\nabla f \cdot D^{s-2}\nabla g$. Moreover, we have the following estimate:

Lemma A.3.1.2. *For any multi-indices α, β one can find a constant $C > 0$ so that for*

$$(\xi,\eta) \in \Gamma = \{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n; \, 0 < |\xi| \le |\eta|/2\},$$

one has the estimate

$$\sup_{0\le\theta\le1} |\partial_\xi^\alpha \partial_\eta^\beta a_s(\xi,\eta,\theta)| \le C|\eta|^{s-|\alpha|-|\beta|}.$$

Lemma A.3.1.2 and the Coifman-Meyer estimate show that for any $f, g \in \mathscr{S}$ which satisfy (A.3.1.5),

$$\|\widetilde{A}_s^\alpha(f,g)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|D^{s-|\alpha|}g\|_{L^{p_2}(\mathbb{R}^n)}.$$

Since

$$\partial_\theta^m a_s(\xi,\eta,\theta) = \sum_{|\alpha|=m} \alpha!\,\partial_\eta^\alpha a_s(\xi,\eta,\theta)\xi^\alpha,$$

we have for any $f, g \in \mathscr{S}$ which satisfy (A.3.1.5),

$$\begin{aligned}\|[D^s, f]g\|_{L^p(\mathbb{R}^n)} &= \|A_s^0(1)(f,g) - A_s^0(0)(f,g)\|_{L^p(\mathbb{R}^n)} \\ &\le \int_0^1 \|A_s^1(\theta)(f,g)\|_{L^p(\mathbb{R}^n)} d\theta \\ &\le \sum_{|\alpha|=1} \int_0^1 \|\tilde{A}_s^\alpha(\theta)(\partial^\alpha f, g)\|_{L^p(\mathbb{R}^n)} d\theta \\ &\le C\|Df\|_{L^{p_1}(\mathbb{R}^n)} \|D^{s-1} g\|_{L^{p_2}(\mathbb{R}^n)},\end{aligned} \tag{A.3.1.10}$$

$$\begin{aligned}\|[D^s, f]g - s\nabla f \cdot D^{s-2}\nabla g\|_{L^p(\mathbb{R}^n)} &= \|A_s^0(1)(f,g) - A_s^0(0)(f,g) - A_s^1(0)(f,g)\|_{L^p(\mathbb{R}^n)} \\ &\le \int_0^1 \|A_s^2(\theta)(f,g)\|_{L^p(\mathbb{R}^n)} d\theta \\ &\le \sum_{|\alpha|=2} \int_0^1 \|\widetilde{A}_s^\alpha(\theta)(\partial^\alpha f, g)\|_{L^p(\mathbb{R}^n)} d\theta \\ &\le C\|D^2 f\|_{L^{p_1}(\mathbb{R}^n)} \|D^{s-2} g\|_{L^{p_2}(\mathbb{R}^n)}.\end{aligned} \tag{A.3.1.11}$$

These estimates and the assumptions (A.3.1.5) explain the redistribution the fractional derivatives, since we have the possibility to replace the right hand sides of the last inequalities of (A.3.1.10) and (A.3.1.11) by $C\|D^{s_1} f\|_{L^{p_1}(\mathbb{R}^n)} \|D^{s_2} g\|_{L^{p_2}(\mathbb{R}^n)}$ and $C\|D^{s_1} f\|_{L^{p_1}(\mathbb{R}^n)} \|D^{s_2} g\|_{L^{p_2}(\mathbb{R}^n)}$, respectively, for any couple (s_1, s_2) of non-negative real numbers with $s_1 + s_2 = s$. For details, see Lemma A.3.2.2.

To state the main results in this article, we introduce the following notation. Let $\Phi \in \mathscr{S}$ be radial function and satisfy $\hat{\Phi} \ge 0$,

$$\operatorname{supp} \hat{\Phi} \subset \{\xi \in \mathbb{R}^n;\ 2^{-1} < |\xi| < 2\}, \qquad \sum_{j\in\mathbb{Z}} \hat{\Phi}(2^{-j}\xi) = 1$$

for all $\xi \in \mathbb{R}^n \backslash \{0\}$, where $\hat{\Phi} = \mathfrak{F}\Phi$ is the Fourier transform of Φ. We define $\Phi_j = \mathfrak{F}^{-1}(\hat{\Phi}(2^{-j}\cdot)) = 2^{jn}\Phi(2^j\cdot)$, $\widetilde{\Phi}_j = \sum_{k=-2}^{2} \Phi_{j+k}$, and $\Psi_j = 1 - \sum_{k>j} \Phi_k$ for $j \in \mathbb{Z}$. For simplicity, we denote $\widetilde{\Phi} = \widetilde{\Phi}_0$ and $\Psi = \Psi_0$. For $f \in \mathscr{S}'$, we define $P_j f$, $P_{\le j} f$, and $P_{>j} f$ as

$$P_j f = \Phi_j * f, \qquad P_{\le j} f = \Psi_j * f, \qquad P_{>j} f = \left(\sum_{k>j} \Phi_k\right) * f,$$

respectively, where $*$ denotes the convolution.

We are ready now to state our main results.

Theorem A.3.1.1. *Let $\ell \in \mathbb{N}$. Let p, p_1, p_2 satisfy $1 < p, p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. Let s, s_1, s_2 satisfy $0 \le s_1, s_2$ and $\ell - 1 \le s = s_1 + s_2 \le \ell$. Then the following bilinear estimate*

$$\begin{aligned}&\left\| D^s(fg) - \sum_{k\in\mathbb{Z}} \sum_{m=0}^{\ell-1} A_s^m(0)(P_{\le k-3} f, P_k g) - \sum_{j\in\mathbb{Z}} \sum_{m=0}^{\ell-1} A_s^m(0)(P_{\le j-3} g, P_j f)\} \right\|_{L^p(\mathbb{R}^n)} \\ &\le C\|D^{s_1} f\|_{L^{p_1}(\mathbb{R}^n)} \|D^{s_2} g\|_{L^{p_2}(\mathbb{R}^n)}\end{aligned}$$

holds for all $f, g \in \mathscr{S}$, where C is a constant depending only on n, p, p_1, p_2.

Moreover, we have the generalization of (A.3.1.2) and simple correction term for $s \ge 2$ as a corollary of Theorem A.3.1.1.

Corollary A.3.1.1. *Let p, p_1, p_2 satisfy $1 < p, p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. Let s, s_1, s_2 satisfy $0 \le s_1, s_2 \le 1$, and $s = s_1 + s_2$. Then the following bilinear estimate*

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^p(\mathbb{R}^n)} \le C\|D^{s_1} f\|_{L^{p_1}(\mathbb{R}^n)} \|D^{s_2} g\|_{L^{p_2}(\mathbb{R}^n)}$$

holds for all $f, g \in \mathscr{S}$.

Corollary A.3.1.2. *Let p, p_1, p_2 satisfy $1 < p, p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. Let s, s_1, s_2 satisfy $0 \le s_1, s_2 \le 2$ and $s = s_1 + s_2 \ge 2$. Then the following bilinear estimate*

$$\|D^s(fg) - fD^s g - gD^s f + sD^{s-2}(\nabla f \cdot \nabla g)\|_{L^p(\mathbb{R}^n)} \le C\|D^{s_1} f\|_{L^{p_1}(\mathbb{R}^n)} \|D^{s_2} g\|_{L^{p_2}(\mathbb{R}^n)}$$

holds for all $f, g \in \mathscr{S}$.

This article is organized as follows. In Section A.3.2, we collect some basic estimates and key estimates for the commutators. In Section A.3.3, we prove Lemma A.3.1.2, Theorem A.3.1.1 and Corollaries A.3.1.1, and A.3.1.2.

A.3.2 Preliminaries

We collect some preliminary estimates needed in the proofs of the main results. For the purpose, we introduce some notations. Let $\mu(p) = \max\{p, (p-1)^{-1}\}$. For $1 \le p \le \infty$, $1 \le q \le \infty$, and $s \in \mathbb{R}$, let $\dot{F}^s_{p,q} = \dot{F}^s_{p,q}(\mathbb{R}^n)$ be the usual homogeneous Triebel-Lizorkin space with

$$\|f\|_{\dot{F}^s_{p,q}} = \|(2^{sj} P_j f)\|_{L^p((\mathbb{R}^n);l^q_j)} = \|\|(2^{sj} P_j f)\|_{l^q_j}\|_{L^p(\mathbb{R}^n)}.$$

It is well known that for $s \in \mathbb{R}$ and $1 < p < \infty$, $\dot{F}^s_{p,2}$ may be identified with $\dot{H}^s_p$, where $\dot{H}^s_p = D^{-s} L^p(\mathbb{R}^n)$ is the usual homogeneous Sobolev space and $\dot{F}^s_{p,q}$ is continuously embedded into $\dot{F}^s_{p,\infty}$. We also define the Hardy-Littlewood maximal operator by

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B(r)|} \int_{B(r)} |f(x+y)| dy,$$

where $B(r) = \{\xi \in \mathbb{R}^n; |\xi| \le r\}$. For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we put $\langle x \rangle = (1 + |x|^2)^{1/2}$, where $|x|^2 = x_1^2 + \cdots + x_n^2$. We adopt the standard multi-index notation such as $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, where $\partial_m = \partial/\partial x_m$, $m = 1, \cdots, n$.

Lemma A.3.2.1 ([39, Theorem 5.1.2])**.** *The estimates*

$$\mu(p)^{-1} \|f\|_{L^p(\mathbb{R}^n)} \le \|f\|_{\dot{F}^0_{p,2}} \le \mu(p) \|f\|_{L^p(\mathbb{R}^n)}$$

hold for $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$.

Lemma A.3.2.2 ([39, Theorem 2.1.10])**.** *Let $s \ge 0$. Then $x \cdot \nabla D^s \Psi \in L^1(\mathbb{R}^n)$. The estimate*

$$|D^s P_{\le k} f(x)| \le 2^{sk} \|x \cdot \nabla D^s \Psi\|_{L^1(\mathbb{R}^n)} Mf(x)$$

holds for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}^n$, where C depends only on n.

proof. For completeness, we give its proof here: Recall that (Ψ_k) and Ψ are radial Schwartz functions satisfying

$$\widehat{P_{\le k} f}(\xi) = \widehat{\Psi}_k(\xi) \widehat{f}(\xi), \quad \widehat{P_{\le 0} f}(\xi) = \widehat{\Psi}(\xi) \widehat{f}(\xi).$$

Using a rescaling argument, combined with the relation

$$D^s P_{\le k} = D^s S^*_{2^k} P_{\le 0} S_{2^k} = 2^{sk} S^*_{2^k} D^s P_{\le 0} S_{2^k}, \quad S^*_{2^k} M S_{2^k} = M,$$

one can reduce the proof of Lemma A.3.2.2 to the case when $k = 0$, where $S_{2^k} f = f(2^{-k}x)$ and $S^*_{2^k} f = f(2^k x)$. Let $\rho \in C^\infty([0,\infty);[0,1])$ satisfy

$$\rho = \begin{cases} 1 & \text{if} \quad 0 \le x \le 1/2, \\ \searrow & \text{if} \quad 1/2 < x < 1, \\ 0 & \text{if} \quad x \ge 1, \end{cases}$$

and $\rho_R(\cdot) = \rho(\cdot/R)$ for any $R > 0$. Let

$$F_x(r) = \int_{S^{n-1}} f(x + r\omega) d\omega, \qquad G_x(r) = \int_0^r F_x(r') r'^{n-1} dr'.$$

Since Ψ and $D^s\Psi$ are radial functions, it is useful to introduce the notation $\psi_s(|\cdot|) = D^s\Psi(\cdot)$. By integration by parts,

$$\begin{aligned} |D^s P_{\le 0} f| &= \lim_{R\to\infty} \left| \int f(x+y)\rho_R(|y|) D^s\Psi(y) dy \right| \\ &= \lim_{R\to\infty} \left| \int_0^R F_x(r) r^{n-1} \rho_R(r)\psi_s(r) dr \right| \\ &= \lim_{R\to\infty} \left| \underbrace{G_x(R)\rho_R(R)\psi_s(R)}_{=0} - \underbrace{G_x(0)\rho_R(0)\psi_s(0)}_{=0} - \int_0^R G_x(r) \frac{d}{dr}(\rho_R \psi_s)(r) dr \right| \\ &\le |S^{n-1}| \int_0^\infty r^{n-1} \left| r \frac{d}{dr}\psi_s(r) \right| dr Mf(x) \\ &= \int_{\mathbb{R}^n} |x \cdot \nabla D^s \Psi(x)| dx \ Mf(x). \end{aligned}$$

Q.E.D.

Remark A.3.2.1. *One can show that* $\|x \cdot \nabla D^s \Psi\|_{L^1(\mathbb{R}^n)}$ *is bounded as follows:*

$$\begin{aligned} \int_{\mathbb{R}^n} |x \cdot \nabla D^s\Psi(x)| dx &= \int_{\mathbb{R}^n} |(n+s) D^s\Psi(x) + D^s \nabla(x\Psi)(x)| dx \\ &\le (n+s)\|D^s\Psi\|_{L^1(\mathbb{R}^n)} + \|D^s\nabla(x\Psi)\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

For any $s \ge 0$,

$$\|D^s\Psi\|_{L^1(\mathbb{R}^n)} \le C\|\Psi\|_{\dot{B}^s_{1,1}} \le C(\|\Psi\|_{\dot{B}^0_{1,\infty}} + \|\Psi\|_{\dot{B}^{2\lceil s/2\rceil}_{1,\infty}}) \le C\|\Psi\|_{H_1^{2\lceil s/2\rceil}},$$

where $\lceil s\rceil = \min\{a \in \mathbb{Z};\, a \ge s\}$. *Moreover, since* $\operatorname{supp} \nabla\hat{\Psi} \subset \mathbb{R}^n \backslash B(1)$, $D^s\nabla(x\Psi) \in \mathscr{S}$ *and* $\|D^s\nabla(x\Psi)\|_{L^p(\mathbb{R}^n)} < \infty$.

Lemma A.3.2.3 ([39, Theorem 2.1.6])**.** *Let* $1 < p \le \infty$ *and* $f \in L^p(\mathbb{R}^n)$. *Then the estimate*

$$\|Mf\|_{L^p(\mathbb{R}^n)} \le 3^{n/p} p' \|f\|_{L^p(\mathbb{R}^n)}$$

holds.

Lemma A.3.2.4 (Fefferman-Stein[32][39, Theorem 1.2])**.** *Let $(f_j)_{j\in\mathbb{Z}}$ be a sequence of measurable functions on $\mathbb{R}^n$. Let $1<p<\infty$ and $1<q\le\infty$. Then the estimate*

$$\|(Mf_j)\|_{L^p((\mathbb{R}^n);l^q_j)} \le C_n\mu(p)\mu(q)\|(f_j)\|_{L^p((\mathbb{R}^n);l^q_j)}$$

holds.

Lemma A.3.2.5. *Let s_1,s_2,s_3,s_4,s_5 be non-negative real numbers satisfying $s_1+s_2+s_3=s_4+s_5$ and let $1<p,p_1,p_2<\infty$ satisfy $1/p=1/p_1+1/p_2$. Then*

$$\left\|D^{s_1}\sum_{j\in\mathbb{Z}}P_jD^{s_2}fP_jD^{s_3}g\right\|_{L^p(\mathbb{R}^n)} \le Cp\mu(p_1)\mu(p_2)\|f\|_{\dot F^{s_4}_{p_1,2}}\|g\|_{\dot F^{s_5}_{p_2,2}}.$$

proof. By the Hölder and Fefferman-Stein inequalities, for any $h\in L^{p'}(\mathbb{R}^n)$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}D^{s_1}\sum_{j\in\mathbb{Z}}P_jD^{s_2}f(x)P_jD^{s_3}g(x)h(x)dx\right|\\
&=\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|(D^{s_1}\Psi_{j+2}(x-y)D^{s_2}P_jf(y)D^{s_3}P_jg(y)h(x)|dydx\\
&\le\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}|D^{s_1}\Psi_{j+2}|*|h|(y)|D^{s_2}P_jf(y)D^{s_3}P_jg(y)|dy\\
&\le Cp\left\|\|2^{s_4j}MP_jf(y)\|_{l^2_j}\|2^{s_5j}MP_jg\|_{l^2_j}\right\|_{L^p(\mathbb{R}^n)}\|h\|_{L^{p'}(\mathbb{R}^n)}\\
&\le Cp\mu(p_1)\mu(p_2)\|h\|_{L^{p'}(\mathbb{R}^n)}\|f\|_{\dot F^{s_1}_{p_1,2}}\|g\|_{\dot F^{s_2}_{p_2,2}}.
\end{aligned}$$

Q.E.D.

Recall the definition of the Hörmander class S^s.

Definition A.3.2.2. *Let $s\in\mathbb{R}$. We say that a symbol*

$$\sigma\in C^\infty(\mathbb{R}^n\setminus\{0\})$$

belongs to the Hörmander class S^s, if for all multi-indices α, we have

$$|\partial^\alpha_\xi\sigma(\xi)|\le C_\alpha|\xi|^{s-|\alpha|},\quad \forall\xi\ne0.$$

Lemma A.3.2.6. *Let $s\ge0$. If $a\in S^s$, then for all multi-indices α,β and (ξ,η) in the cone Γ, defined in* (A.3.1.6)*, we have*

$$\sup_{0\le\theta\le1}|\partial^\alpha_\xi\partial^\beta_\eta a(\eta+\theta\xi)|\le C_{\alpha+\beta}|\eta|^{s-|\alpha|-|\beta|}.$$

proof. For $\alpha,\beta\in\mathbb{N}^n_0$,

$$\partial^\alpha_\xi\partial^\beta_\eta a(\eta+\theta\xi)=(\partial^{\alpha+\beta}_\eta a)(\eta+\theta\xi)\theta^{|\alpha|}.$$

and for $(\xi,\eta)\in\Gamma$,

$$\frac12|\eta|\le|\eta+\theta\xi|\le\frac32|\eta|.$$

The required estimate is established and the proof is complete. Q.E.D.

A.3.3 Proofs of Lemma A.3.1.2, Theorem A.3.1.1, and Corollaries A.3.1.1 and A.3.1.2.

A.3.3.1 Proof of Lemma A.3.1.2 and Theorem A.3.1.1

Proof of Lemma A.3.1.2. Since $|\cdot|^s \in S^s$ and Lemma A.3.2.6, we are done. Q.E.D.

To prove Theorem A.3.1.1, Corollaries A.3.1.1 and A.3.1.2, we introduce the following notation. For bilinear operator B, defined by

$$B(F,G)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi+\eta)} b(\xi,\eta)\widehat{F}(\xi)\widehat{G}(\eta)d\xi d\eta,$$

we can define

$$B_{\ll}(f,g) = \sum_{k\in\mathbb{Z}} B(P_{\leq k-3}f, P_k g), \quad B_{\sim}(f,g) = \sum_{j\in\mathbb{Z}}\sum_{k=j-2}^{j+2} B(P_j f, P_k g).$$

Obviously, we have the decomposition

$$B(f,g) = B_{\ll}(f,g) + B_{\sim}(f,g) + B_{\ll}(g,f) \tag{A.3.3.1}$$

and the symbol $b_{\ll}(\xi,\eta)$ of $B_{\ll}$ is defined by

$$b_{\ll}(\xi,\eta) = \sum_{k\in\mathbb{Z}} \widehat{\Psi_{k-3}}(\xi)\widehat{\Phi_k}(\eta)b(\xi,\eta). \tag{A.3.3.2}$$

We have the following useful property.

Lemma A.3.3.1. *Let $s \geq k \geq 0$ and s_1, s_2 are non-negative real numbers satisfying*

$$s_1 \leq k, \;\; s_1 + s_2 = s$$

and let $1 < p, p_1, p_2 < \infty$ satisfy $1/p = 1/p_1 + 1/p_2$. Then the bilinear form $B_{\ll}(f,g)$ with symbol of type (A.3.3.2) *with b in the Coifman - Meyer class satisfies*

$$\|B_{\ll}(D^k f, D^{s-k} g)\|_{L^p(\mathbb{R}^n)} \leq C\|D^{s_1} f\|_{L^{p_1}(\mathbb{R}^n)}\|D^{s_2} g\|_{L^{p_2}(\mathbb{R}^n)}.$$

The proof follows from the Coifman - Meyer estimate (A.3.1.7) and we skip it.
Lemma A.3.2.6 implies:

Lemma A.3.3.2. *Let $s \geq 0$. If $a \in S^s(\mathbb{R}^n)$, then with $a^s(\xi,\eta,\theta) = |\eta + \theta\xi|^s$ we have*

$$\sup_{0\leq\theta\leq 1} |\partial_\xi^\alpha \partial_\eta^\beta a^s_{\ll}(\xi,\eta,\theta)| \leq C|\eta|^{s-|\alpha|-|\beta|}.$$

Another useful application of the Coifman - Meyer estimate (A.3.1.7) concerns the bilinear form

$$B(f,g) = D^{s_1}(D^{s_2} f D^{s_3} g). \tag{A.3.3.3}$$

Lemma A.3.3.3. *Let s_1, s_2, s_3, s_4, s_5 be non-negative numbers satisfying*

$$s_1 + s_2 + s_3 = s_4 + s_5, \;\; s_4 \leq s_2$$

and let $1 < p, p_1, p_2 < \infty$ *satisfy* $1/p = 1/p_1 + 1/p_2$. *Then the bilinear form* (A.3.3.3) *satisfies*

$$\|B_{\ll}(f,g)\|_{L^p(\mathbb{R}^n)} \leq C\|D^{s_4} f\|_{L^{p_1}(\mathbb{R}^n)} \|D^{s_5} g\|_{L^{p_2}(\mathbb{R}^n)}.$$

Proof of Theorem A.3.1.1. Consider the bilinear form $B(f,g) = D^s(fg)$. We have the decomposition (A.3.3.1). For the term $B_{\sim}(f,g)$ we can apply the estimate of Lemma A.3.2.5. Therefore, it is sufficient to show that

$$B_{\ll}(f,g) = \sum_{m=0}^{\ell-1} A^m_{s,\ll}(0)(f,g) + \sum_{|\alpha|=\ell} T^\alpha_{\ll}(\partial^\alpha f, D^{s-\ell} g), \tag{A.3.3.4}$$

where $T^\alpha_{\ll}$ is a Coifman-Meyer bilinear form

$$T^\alpha_{\ll}(F,G)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi+\eta)} t^\alpha_{\ll}(\xi,\eta) \widehat{F}(\xi)\widehat{G}(\eta) d\xi d\eta$$

with symbol $t^\alpha_{\ll}(\xi,\eta)$ in the CM class supported in $\{|\xi| \leq |\eta|/2\}$, so it satisfies the estimate

$$\|T^\alpha_{\ll}(F,G)\|_{L^p(\mathbb{R}^n)} \leq C_{p,p_1,p_2}\|F\|_{L^{p_1}(\mathbb{R}^n)}\|G\|_{L^{p_2}(\mathbb{R}^n)} \tag{A.3.3.5}$$

for all $1 < p, p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$.

We can use the Taylor expansion with respect to θ:

$$a_s(\xi,\eta,1) = \sum_{m=0}^{\ell-1} \frac{1}{m!}\partial^m_\theta a_s(\xi,\eta,0) + \frac{1}{(\ell-1)!}\int_0^1 (1-\theta)^{\ell-1}\partial^\ell_\theta a_s(\xi,\eta,\theta) d\theta$$

and note that (A.3.1.9) implies

$$B_{\ll}(f,g) = A^0_{s,\ll}(1)(f,g)$$

so the Taylor expansion for $a_s(\xi,\eta,1)$ implies (A.3.3.4) with symbol

$$t^\alpha_{\ll}(\xi,\eta) = \sum_{k\in\mathbb{Z}} \widehat{\Psi_{k-3}}(\xi)\widehat{\Phi}_k(\eta)\int_0^1 (1-\theta)^{|\alpha|-1}\theta^{|\alpha|}\partial^\alpha_\eta a_s(\xi,\eta,\theta)\frac{d\theta}{(|\alpha|-1)!}|\eta|^{-s+|\alpha|}.$$

An application of Lemma A.3.3.2 shows that $t^\alpha_{\ll}(\xi,\eta)$ belongs to the CM class so the Coifman-Meyer estimate proves (A.3.3.5) and completes the proof of the theorem. Q.E.D.

A.3.3.2 Proof of Corollary A.3.1.1

Let $B(f,g) = D^s(fg) - fD^s g - gD^s f$. Then the term $B_{\sim}(f,g)$ can be estimated by Lemma A.3.2.5. So it is sufficient to check the estimate

$$\|B_{\ll}(f,g)\|_{L^p(\mathbb{R}^n)} \leq C\|D^{s_1} f\|_{L^{p_1}(\mathbb{R}^n)}\|D^{s_2} g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{A.3.3.6}$$

The term $B_{\ll}(f,g)$ can be represented as

$$B_{\ll}(f,g) = B^I_{\ll}(f,g) + B^{II}_{\ll}(f,g),$$

where

$$B^I(f,g) = D^s(fg) - fD^s g$$

and

$$B^{II}(f,g) = -gD^s f.$$

The symbol of

$$B^I_{\ll}(f,g) = A^0_s(1)(P_{\le k-3}f, P_k g) - A^0_s(0)(P_{\le k-3}f, P_k g),$$

can be represented as by the aid of the Taylor expansion

$$a_s(\xi,\eta,1) - a_s(\xi,\eta,0) = \int_0^1 \partial_\theta a_s(\xi,\eta,\theta)d\theta$$

so as in (A.3.3.4) we have

$$B^I_{\ll}(f,g) = \sum_{|\alpha|=1} T^\alpha_{\ll}(\partial^\alpha f, D^{s-1}g)$$

with symbol

$$t^\alpha_{\ll}(\xi,\eta) = \sum_{k\in\mathbb{Z}} \widehat{\Psi_{k-3}}(\xi)\widehat{\Phi_k}(\eta) \int_0^1 \theta\partial^\alpha_\eta a(\xi,\eta,\theta)d\theta|\eta|^{-s+1}$$

in the CM class. Applying Lemma A.3.3.1, we get

$$\|B^I_{\ll}(f,g)\|_{L^p(\mathbb{R}^n)} \le C\|D^{s_1}f\|_{L^{p_1}(\mathbb{R}^n)}\|D^{s_2}g\|_{L^{p_2}(\mathbb{R}^n)}.$$

The term $B^{II}_{\ll}(f,g)$ can be estimated by the aid of Lemma A.3.3.1 again, so we get (A.3.3.6) and the proof is complete.

A.3.3.3 Proof of Corollary A.3.1.2

Let $B(f,g) = D^s(fg) - fD^s g - gD^s f + sD^{s-2}(\nabla f\cdot\nabla g)$. The term $B_{\sim}(f,g)$ can be estimated by using Lemma A.3.2.5. As in the proof of Corollary A.3.1.1, it is sufficient to show

$$\|B_{\ll}(f,g)\|_{L^p(\mathbb{R}^n)} \le C\|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{A.3.3.7}$$

The term $B_{\ll}(f,g)$ can be represented as follows

$$B_{\ll}(f,g) = B^I_{\ll}(f,g) + B^{II}_{\ll}(f,g) + B^{III}_{\ll}(f,g),$$

where

$$\begin{aligned} B^I(f,g) &= D^s(fg) - fD^s g + s\nabla f\cdot D^{s-2}\nabla g,\\ B^{II}(f,g) &= sD^{s-2}(\nabla f\cdot\nabla g) - s\nabla f\cdot D^{s-2}\nabla g,\\ B^{III}(f,g) &= -gD^s f. \end{aligned}$$

Then

$$\begin{aligned} B^I_{\ll}(f,g) &= A^0_{s,\ll}(1)(f,g) - A^0_{s,\ll}(0)(f,g) - A^1_{s,\ll}(0)(f,g) = \sum_{|\alpha|=2} T^\alpha_{\ll}(\partial^\alpha f, D^{s-2}g),\\ B^{II}_{\ll}(f,g) &= \sum_{m=1}^n s\{A^0_{s-2,\ll}(1)(\partial_m f,\partial_m g) - A^0_{s-2,\ll}(0)(\partial_m f,\partial_m g)\}\\ &= \sum_{m=1}^n\sum_{|\alpha|=1} s\widetilde{T}^\alpha_{\ll}(\partial^\alpha\partial_m f, D^{s-3}\partial_m g) \end{aligned}$$

with symbol

$$t^{\alpha}_{\ll}(\xi,\eta) = \sum_{k\in\mathbb{Z}} \widehat{\Psi_{k-3}}(\xi)\widehat{\Phi_k}(\eta)\int_0^1 (1-\theta)\theta^2\partial_\eta^\alpha a_s(\xi,\eta,\theta)d\theta|\eta|^{-s+2},$$
$$\widetilde{t}^{\alpha}_{\ll}(\xi,\eta) = \sum_{k\in\mathbb{Z}} \widehat{\Psi_{k-3}}(\xi)\widehat{\Phi_k}(\eta)\int_0^1 \theta\partial_\eta^\alpha a_{s-2}(\xi,\eta,\theta)d\theta|\eta|^{-s+3}$$

in the CM class. Applying Lemma A.3.3.1, we can estimate $B^{I}_{\ll}(f,g)$, $B^{II}_{\ll}(f,g)$ and $B^{III}_{\ll}(f,g)$ and deduce (A.3.3.7).

This completes the proof.

Bibliography

[1] H. Ando, T. Horiuchi, and E. Nakai, "Construction of slowly increasing functions," Sci. Math. Japonicae Online, **e-2012**(2012), 207-221.

[2] H. Bahouri, J.-Y. Chemin, and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Springer, 2011.

[3] I. Bejenaru and T. Tao, "Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation," J. Funct. Anal., **233**(2006), 228–259.

[4] Á. Bényi and T. Oh, "On a class of bilinear pseudodifferential operators," J. Funct. Spaces Appl., (2013), Art. ID 560976, 5.

[5] Á. Bényi and T. Oh, "Smoothing of commutators for a Hörmander class of bilinear pseudodifferential operators," J. Fourier Anal. Appl., **20**(2014), 282–300.

[6] J. Bergh, and J. Löfström, Interpolation spaces. An introduction, Springer-Verlag, Berlin-New York, 1976.

[7] F. Bernicot, D. Maldonado, K. Moen, and V. Naibo, "Bilinear Sobolev-Poincaré inequalities and Leibniz-type rules," J. Geom. Anal., **24**(2014), 1144–1180.

[8] J. L. Bona, G. Ponce, J.-C. Saut, and C. Sparber, "Dispersive blow-up for nonlinear Schrödinger equations revisited," J. Math. Pures Appl., **102**(2014), 782–811.

[9] J. Bourgain, "Exponential sums and nonlinear Schrödinger equations," Geom. Funct. Anal., **3**(1993), 157–178.

[10] J. Bourgain, "Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations," Geom. Funct. Anal., **3**(1993), 107–156.

[11] J. Bourgain. "Nonlinear Schrödinger equations," In *Hyperbolic equations and frequency interactions (Park City, UT, 1995)*, volume 5 of *IAS/Park City Math. Ser.*, 3–157. Amer. Math. Soc., Providence, RI, (1999).

[12] J. Bourgain and D. Li, "On an endpoint Kato-Ponce inequality," Differential Integral Equations, **27**(2014), 1037–1072.

[13] N. Bournaveas, "Local well-posedness for a nonlinear Dirac equation in spaces of almost critical dimension," Discrete Contin. Dyn. Syst., **20**(2008) 605–616.

[14] N. Bournaveas, T. Candy, and S. Machihara, "Local and global well posedness for the Chern-Simons-Dirac system in one dimension," Differential Integral Equations, **25**(2012), 699–718.

[15] J. P. Borgna and D. F. Rial, "Existence of ground states for a one-dimensional relativistic Schrödinger equation," J. Math. Phys., **53**(2012), 062301.

[16] H. Brézis, Functional analysis, Sobolev spaces and partial differential equations, Springer, New York, 2011.

[17] H. Brézis and T. Gallouet, "Nonlinear Schrödinger evolution equations," Nonlinear Anal., **4**(1980), 677–681.

[18] D. Cai, A. J. Majda, D. W. McLaughlin, and E. G. Tabak, "Dispersive wave turbulence in one dimension," Phys. D, **152/153**(2001), 551–572.

[19] A.-P. Calderón, "Commutators of singular integral operators," Proc. Nat. Acad. Sci. U.S.A., **53**(1965), 1092–1099.

[20] T. Cazenave, Semilinear Schrödinger equations, American Mathematical Society, 2003.

[21] T. Cazenave and F. B. Weissler, Some remarks on the nonlinear Schrödinger equation in the subcritical case, In *New methods and results in nonlinear field equations (Bielefeld, 1987)*, volume 347 of *Lecture Notes in Phys.*, 59 - 69, Springer, Berlin, (1989).

[22] Y. Cho and T. Ozawa, "On the semirelativistic Hartree-type equation," SIAM J. Math. Anal., **38**(2006), 1060–1074.

[23] F. M. Christ and M. I. Weinstein, "Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation," J. Funct. Anal., **100**(1991), 87–109.

[24] R. Coifman and Y. Meyer, "On commutators of singular integrals and bilinear singular integrals," Trans. Amer. Math. Soc., **212**(1975), 315–331.

[25] R. Coifman and Y. Meyer, "Commutateurs d'intégrales singulières et opérateurs multilinéaires," Ann. Inst. Fourier (Grenoble), **28**(1978), xi, 177–202.

[26] R. Coifman and Y. Meyer, Ondelettes et opérateurs. III, Hermann, Paris, (1991).

[27] S. Cuccagna, V. Georgiev, and N. Visciglia, "Decay and scattering of small solutions of pure power NLS in $\mathbb{R}$ with $p > 3$ and with a potential", Comm. Pure Appl. Math., **67**(2014), 957–981.

[28] P. D'Ancona, D. Foschi, and S. Selberg, "Atlas of products for wave-Sobolev space on $\mathbb{R}^{1+3}$," Trans. Amer. Math. Soc., **364**(2012), 31–63.

[29] A. Elgart and B. Schlein, "Mean field dynamics of boson stars," Commun. Pure Appl. Math., **60**(2007), 500–545.

[30] M. Escobedo and L. Vega, "A semilinear Dirac equation in $H^s(\mathbb{R}^3)$ for $s > 1$," SIAM J. Math. Anal., **28**(1997), 338–362.

[31] J. Fan and T. Ozawa, "Regularity criterion for weak solutions to the Navier-Stokes equations in terms of the gradient of the pressure," J. Inequal. Appl., **2008**, Art. ID 412678, 6pp.

[32] C. Fefferman and E. M. Stein, "Some maximal inequalities," Amer. J. Math., **93**(1971), 107–115.

[33] G. B. Folland, Real analysis, John Wiley & Sons, Inc., New York, 1999.

[34] A. E. Gatto, "Product rule and chain rule estimates for fractional derivatives on spaces that satisfy the doubling condition," J. Funct. Anal., **188**(2002), 27–37.

[35] J. Ginibre, T. Ozawa, and G. Velo, "On the existence of the wave operators for a class of nonlinear Schrödinger equations," Ann. Inst. H. Pincaré Pys. Théor., **60**(1994), 211–239.

[36] J. Ginibre, Y. Tsutsumi, and G. Velo, "On the Cauchy problem for the Zakharov system," J. Funct. Anal., **151**(1997), 384–436.

[37] L. Grafakos and Z. Si, "The Hörmander multiplier theorem for multilinear operators," J. Reine Angew. Math., **668**(2012), 133–147.

[38] L. Grafakos, Classical Fourier analysis, Springer, New York, (2008).

[39] L. Grafakos. "Multilinear operators in harmonic analysis and partial differential equations," RIMS Kôkyûroku Bessatsu, **B33**(2012), 11–27.

[40] L. Grafakos, L. Liu, and D. Yang, "Vector-valued singular integrals and maximal functions on spaces of homogeneous type," Math. Scand., **104**(2009), 296–310.

[41] L. Grafakos, D. Maldonado, and V. Naibo, "A remark on an endpoint Kato-Ponce inequality," Differential Integral Equations, **27**(2014), 415–424.

[42] L. Grafakos, A. Miyachi, and N. Tomita, "On multilinear Fourier multipliers of limited smoothness," Canad. J. Math., **65**(2013), 299–330.

[43] L. Grafakos and S. Oh, "The Kato-Ponce Inequality," Comm. Partial Differential Equations, **39**(2014), 1128–1157.

[44] L. Grafakos and Z. Si, "The Hörmander multiplier theorem for multilinear operators," J. Reine Angew. Math., **668**(2012), 133–147.

[45] L. Grafakos and R. H. Torres, "Multilinear Calderón-Zygmund theory," Adv. Math., **165**(2002), 124–164.

[46] B. Guo and D. Huang, "Existence and stability of standing waves for nonlinear fractional Schrödinger equations," J. Math. Phys., **53**(2012), 083702, 15.

[47] Z. Guo, Y. Lin, and L. Molinet, "Well-posedness in energy space for the periodic modified Banjamin-Ono equation," J. Differential Equations, **256**(2014), 2778–2806.

[48] N. Hayashi, C. Li, and P. I. Naumkin, "On a system of nonlinear Schrödinger equations in 2d," Differential Integral Equations, **24**(2011), 417–434.

[49] N. Hayashi, C. Li, and T. Ozawa, "Small data scattering for a system of nonlinear Schrödinger equations," Differ. Equ. Appl., **3**(2011), 415–426.

[50] N. Hayashi, T. Ozawa, and K. Tanaka, "On a system of nonlinear Schrödinger equations with quadratic interaction," Ann. I. H. Poincaré, **30**(2013), 661–690.

[51] G. Hoshino and T. Ozawa, "Analytic smoothing effect for a system of nonlinear Schrödinger equations," to appear in Differ. Equ. Appl., **5**(2013), 395–408.

[52] M. Ikeda and T. Inui, "Some non-existence results for the semilinear Schrödinger equation without gauge invariance," J. Math. Anal. Appl., **425**(2015), 758–773.

[53] M. Ikeda and T. Inui, "Small data blow up of L^2 or H^1-solution for the semilinear Schrödinger equation without gauge invariance," J. Evol. Equ., **15**(2015), 571–581.

[54] M. Ikeda and Y. Wakasugi, "Small-data blow-up of L^2-solution for the nonlinear Schrödinger equation without gauge invariance," Differential Integral Equations, **26**(2013), 1275–1285.

[55] T. Inui, "Some nonexistence results for a semirelativistic Schrödinger equation with nongauge power type nonlinearity," Proc. Amer. Math. Soc., **144**(2016), 2901–2909.

[56] T. Kato, "On nonlinear Schrödinger equations II. H^s solutions and unconditional well-posedness," J. Anal. Math., **67**(1995), 281–306.

[57] T. Kato and G. Ponce, "Commutator estimates and the Euler and Navier-Stokes equations," Comm. Pure Appl. Math., **41**(1988), 891–907.

[58] C. E. Kenig, G. Ponce, and L. Vega, "Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle," Comm. Pure Appl. Math., **46**(1993), 527–620.

[59] C. E. Kenig and E. M. Stein, "Multilinear estimates and fractional integration," Math. Res. Lett., **6**(1999), 1–15.

[60] S. Klainerman and S. Selberg, "Bilinear estimates and applications to nonlinear wave equations," Comm. Contemp. Math., **4**, (2002), 223–295.

[61] H. Kozono and Y. Taniuchi, "Bilinear estimates in BMO and the Navier-Stokes equations," Math. Z., **235**, (2000), 173–194.

[62] J. Krieger, E. Lenzmann, and P. Raphaël, "Nondispersive solutions to the L^2-critical half-wave equation," Arch. Ration. Mech. Anal., **209**(2013), 61–129.

[63] N. Laskin, "Fractional quantum mechanics and Lévy integral," Phys. Lett. A, **268**(2000), 298–305.

[64] N. Laskin, "Fractional quantum mechanics," Phys. Rev. E, **62**(2000), 3135.

[65] N. Laskin, "Fractional Schrödinger equations," Phys. Rev. E, **66**(2002), 056108.

[66] E. Lenzmann, "Well-posedness for semi-relativistic Hartree equations of critical type," Math. Phys. Anal. Geom., **10**(2007), 43–64.

[67] E. H. Lieb and H. T. Yau, "The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics," Commun. Math. Phys., **112**(1987), 147 – 174.

[68] F. Linares, D. Pilod, and J.-C. Saut, "Dispersive perturbations of Burgers and hyperbolic equations I: Local theory," SIAM J. Math. Anal., **46**(2014), 1505–1537.

[69] F. Linares and G. Ponce, Introduction to Nonlinear Dispersive Equations, Springer, 2009.

[70] S. Machihara, M. Nakamura, K. Nakanishi, and T. Ozawa, "Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation," J. Funct. Anal., **219**(2005), 1–20.

[71] S. Machihara, M. Nakamura, and T. Ozawa, "Small global solutions for nonlinear Dirac equations," Differential Integral Equations, **17**(2004) 623–636.

[72] S. Machihara, K. Nakanishi, and T. Ozawa, "Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation," Rev. Mat. Iberoamericana, **19**(2003), 179–194.

[73] S. Machihara, K. Nakanishi, and K. Tsugawa, "Well-posedness for nonlinear Dirac equations in one dimension," Kyoto J. Math., **50**(2010), 403–451.

[74] A. J. Majda, D. W. McLaughlin, and E. G. Tabak, "A one-dimensional model for dispersive wave turbulence," J. Nonlinear Sci., **7**(1997), 9–44.

[75] A. Miyachi and N. Tomita, "Calderón-Vaillancourt-type theorem for bilinear operators," Indiana Univ. Math. J., **62**(2013), 1165–1201.

[76] T. Ogawa, "A proof of Trudinger's inequality and its application to nonlinear Schrödinger equations," Nonlinear Anal., **14**(1990) 765–769.

[77] T. Ogawa and T. Ozawa, "Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrödinger mixed problem," J. Math. Anal. Appl., **155**(1991), 531–540.

[78] T. Oh, "A blowup result for the periodic NLS without gauge invariance," C. R. Math. Acad. Sci. Paris, **350**(2012), 389–392.

[79] T. Ozawa, "Remarks on proofs of conservation laws for nonlinear Schrödinger equations," Calc. Var. Partial Differential Equations, **25**(2006), 403–408.

[80] T. Ozawa and Y. Tsutsumi, "Space-time estimates for null gauge forms and nonlinear Schrödinger equations," Differential Integral Equations, **11**(1998), 201–222.

[81] T. Ozawa and N. Visciglia, "An improvement on the Brézis–Gallouët technique for 2D NLS and 1D half-wave equation," Ann. I. H. Poincaré Anal. Non Linéaire, 2015, http://dx.doi.org/10.1016/j.anihpc.2015.03.004.

[82] T. Ozawa and J. Zhai, "Global existence of small classical solutions to nonlinear Schrödinger equations," Ann. Inst. H. Poincaré Anal. Non Linéaire, **25**(2008), 303–311.

[83] T. Runst and W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, Walter de Gruyter, 1996.

[84] H. Sasaki, $H^{n/2}$に属する非有界関数(ver 1.04), http://www.math.s.chiba-u.ac.jp/s̃asaki/008_sobolev.pdf

[85] J.-C. Saut and R. Temam, "Remarks on the Korteweg-de-Vries equation," Israel J. Math., **2014**(1976), 78–87.

[86] S. Selberg and A. Tesfahun, "Low regularity well-posedness of the Dirac-Klein-Gordon equations in one space dimension," Commun. Contemp. Math., **10**(2008), 181–194.

[87] W. Sickel and H. Triebel, "Hölder inequalities and sharp embeddings in function spaces of B^s_{pq} and F^s_{pq} type," Z. Anal. Anwendungen, **14**(1995), 105–140.

[88] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, N.J., 1970.

[89] T. Tao, "Multilinear weighted convolutions of L^2 functions, and applications to nonlinear dispersive equations," Amer. J. Math., **123** (2001), 839–908.

[90] T. Tao, Nonlinear Dispersive Equations, Local and Global Analysis, Amer. Math. Soc., 2006.

[91] N. Tomita, "A Hörmander type multiplier theorem for multilinear operators," J. Funct. Anal., **259**(2010), 2028–2044.

[92] M. V. Vladimirov, "On the solvability of a mixed problem for a nonlinear equation of Schrödinger type," Dokl. Akad. Nauk SSSR, **275**(1984), 780–783.

[93] Q. S. Zhang, "Blow-up results for nonlinear parabolic equations on manifolds," Duke Math. J., **97**(1999), 515–539.

[94] Q. S. Zhang, "A blow-up result for a nonlinear wave equation with damping: the critical case," C. R. Acad. Sci. Paris Sér. I Math., **333**(2001), 109–114.

Acknowledgments

I would like to offer my sincerest gratitude to my thesis adviser Professor Tohru Ozawa for his constructive and thoughtful comments and continuous support of my Ph.D. study and research.

I owe my deep gratitude to Professor Hideo Kozono, Professor Kazunaga Tanaka, and Professor Mitsuharu Ôtani for their helpful comments and critical reading of the manuscript as the member of the doctoral committee.

I greatly appreciate Professor Vladimir Georgiev for his great hospitality during my stay in Pisa and his insightful comments and suggestions. I would like to offer my special thanks to Professor Shuji Machiara for his kind encouragement and helpful advice. I am deeply grateful to Professor Yoshihiro Shibata for his warm encouragement and financial support via Top Global University project. I would like to show my appreciation to Professor Tokio Matsuyama for his genial encouragement. I owe much to Professor Takahisa Harayama for his gentle encouragement and critical support for the procedure of earning a doctoral degree.

I am indebted to Shun Uchida for his considerable support. I also thank Masayuki Hayashi for his kind help. I would like to express my gratitude to Natsuko Ikezaki for her generous support for the paperwork for my stay in Pisa.

This research was supported by Grant-in-Aid for JSPS Fellows (grant number 26-7371 and 16J30008) and Mathematics and Physics Unit "Multiscale Analysis, Modeling and Simulation" of Top Global University Project in Waseda University.

Finally, I thank my family for their heartily support.

List of Papers

- K. Fujiwara and T. Ozawa, "Exact remainder formula for the Young inequality and applications," Int. J. Math. Anal., **7** (2013), 2723–2735, doi:10.12988/ijma.2013.39230.
- K. Fujiwara and T. Ozawa, "Stability of the Young and Hölder inequalities," J. Inequal. Appl., 2014:162, doi:10.1186/1029-242X-2014-162.
- K. Fujiwara and T. Ozawa, "Identities for the difference between the arithmetic and geometric means," Int. J. Math. Anal., **8** (2014), 1525-1542, doi:10.12988/ijma.2014.46170.
- K. Fujiwara, S. Machihara, and T. Ozawa, "Well-posedness for the Cauchy problem a system of semirelativistic equations," Comm. Math. Phys., **338**(2015), 367–391, doi:10.1007/s00220-015-2347-3.
- K. Fujiwara and T. Ozawa, "On a system of semirelativistic equations in the energy space," Commun. Pure Appl. Anal., **14** (2015), 1343–1355, doi:10.3934/cpaa.2015.14.1343.
- K. Fujiwara, S. Machihara, and T. Ozawa, "Remark on a semirelativistic equation in the energy space," Discrete Contin. Dyn. Syst. Suppl., 2015, 473–478, doi:10.3934/proc.2015.0473.
- K. Fujiwara, "Remark on local solvability of the Cauchy problem for semirelativistic equations," J. Math. Anal. Appl., **432** (2015) 744–748, doi:10.1016/j.jmaa.2015.07.009.
- K. Fujiwara and T. Ozawa, "Remarks on global solutions to the Cauchy problem for semirelativistic equations with power type nonlinearity," Int. J. Math. Anal., **9**(2015), 2599-2610, doi:10.12988/ijma.2015.58211.
- K. Fujiwara and T. Ozawa, "Weighted L^p-boundedness of convolution type integral operators associated with bilinear estimates in the Sobolev spaces," J. Math. Soc. Japan, **68**(2016), 169 – 191, doi:10.2969/jmsj/06810169.
- K. Fujiwara and T. Ozawa, "Remarks on bilinear estimates in the Sobolev spaces," RIMS kôkyûroku Bessatsu, in press.

平成28年7月16日　小澤徹
H.28.7.16　大谷光泰
平成28年7月20日　小薗英雄
H.28.7.16　田中和永